A Cluster-Theoretic Approach to Polynomial Equations (II)

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2002, Cluster algebra

Fomin and Zelevinsky abstracted out an algebraic structure.

A **seed** is a pair $\Omega := (\tilde{B}, \tilde{\mathbf{x}})$, where

- Exchange matrix $\tilde{B} = (b_{ij})$ is an $(n+m) \times n$ integer skew-symmetrizable matrix;
- The cluster $\tilde{\mathbf{x}} = \mathbf{x}_{ex} \cup \mathbf{x}_{fr}$
  where the set of cluster variables $\mathbf{x}_{ex} = \{x_1, \ldots, x_n\}$ and the set of frozen variables $\mathbf{x}_{fr} = \{x_{n+1}, \ldots, x_{n+m}\}$.

The **mutation** of the seed Ω **at direction** $k \in [1, n]$ is defined to be the new seed

$$\mu_k(\tilde{B}, \tilde{\mathbf{x}}) := (\tilde{B}', \tilde{\mathbf{x}}')$$

given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or} j = k \\ b_{ij} + sgn(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise} \end{cases} \quad (1)$$

$$x'_i = \begin{cases} x_i & \text{if } i \neq k \\ \dfrac{P_k^+ \prod\limits_{j \in \langle n \rangle} x_j^{[b_{jk}]_+} + P_k^- \prod\limits_{j \in \langle n \rangle} x_j^{[-b_{jk}]_+}}{x_k} & \text{if } i = k \end{cases} \quad (2)$$

where P_k^+, P_k^- are monomials on $\mathbf{x}_{fr}$.

Then the **cluster algebra** $\mathcal{A}(\Omega)$ is the $R[\mathbf{x}_{fr}]$-subalgebra of the rational function field $\mathcal{F}$ over a commutative ring R generated by all cluster variables $\bigcup \mathbf{x}_{ex}$ of all seeds obtained from the initial seed by any finite steps of mutations.

The case $R = \mathbb{Z}$ gives the classical cluster theory,
write as a $\mathbb{Z}$-cluster algebra $\mathcal{A}(\Omega)_{\mathbb{Z}}$;
For case $R = F_q$ a finite field with $q = p^s$, we give the cluster theory over a finite field, write as a F_q-cluster algebra $\mathcal{A}(\Omega)_{F_q}$.

$\mathbb{Z}$-cluster algebras and F_q-cluster algebras

It is easy to see most of facts for F_q-cluster algebras are the same as for $\mathbb{Z}$-cluster algebras. For example,

Theorem 1 (Laurent phenomenon, (FZ, 2002; L-Pan, 2025))

Given a seed $\Omega := (\tilde{B}, \tilde{\mathbf{x}})$. *Let* $\mathbf{x}' := \mu_{s_m} \cdots \mu_{s_1}(\mathbf{x})$. *Then for all* $i \in [1, n]$, *we have* $x_i' \in R[\mathbf{x}^{\pm}]$ *for either* $R = \mathbb{Z}$ *or* $R = F_q$.

The positivity of cluster variables of cluster algebras holds only for $\mathbb{Z}$-cluster algebras, that is,

Theorem 2 (Positivity of cluster variables, (GHKK, 2018))

Given a seed $\Omega := (\tilde{B}, \tilde{\mathbf{x}})$. *Let* $\mathbf{x}' := \mu_{s_m} \cdots \mu_{s_1}(\mathbf{x})$. *Then for all* $i \in [1, n]$, *we have* $x_i' \in \mathbb{Z}_{\geq 0}[\mathbf{x}^{\pm}]$ *in the* $\mathbb{Z}$*-cluster algebra* $\mathcal{A}(\Omega)_{\mathbb{Z}}$.

Obviously, positivity of cluster variables is NOT necessary to be considered for the F_q-cluster algebra $\mathcal{A}(\Omega)_{F_q}$.

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1880, Markov equation[1]

- Diophantine equation
$$M(X, Y, Z) := X^2 + Y^2 + Z^2 - 3XYZ = 0.$$
- Transformations

$$\begin{aligned} m_1(X, Y, Z) &:= (3YZ - X, Y, Z), \\ m_2(X, Y, Z) &:= (X, 3XZ - Y, Z), \\ m_3(X, Y, Z) &:= (X, Y, 3XY - Z). \end{aligned}$$

- Set of solutions

$$\langle m_1, m_2, m_3 \rangle (1, 1, 1) = \mathcal{V}_{\mathbb{Z}_{>0}}(M)$$

Note that $(1, 1, 1)$ is the trivial solution of $M(X, Y, Z)$.

[1] A. Markoff. Sur les formes quadratiques binaires ind é finies. Math. Ann., 17(3):379 – 399, 1880.

What is the relationship between cluster algebras and number theory?

- Given a cluster algebra $\mathcal{A}(B,(x_1,x_2,x_3))$ where

$$B := \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$

- For $(x_0,y_0,z_0) \in \mathbb{Z}^3_{>0}$ a solution of Markov equation:

$$x^2+y^2+z^2 = 3xyz.$$

  Then $\mu_i(x_0,y_0,z_0) = m_i(x_0,y_0,z_0)$ is also a solution for $i = 1,2,3$. For example,
  $\mu_1(x,y,z) = (\frac{y^2+z^2}{x}, y, z) = (3yz - x, y, z) = m_1(x,y,z)$

- Due to the positivity of cluster variables in $\mathcal{A}(\Omega)_{\mathbb{Z}}$, we have $\mu_i(x_0,y_0,z_0) \in \mathbb{Z}^3_{>0}$. It follows

$$\langle \mu_1,\mu_2,\mu_3 \rangle (1,1,1) = \mathcal{V}_{\mathbb{Z}_{>0}}(M) \text{ the set of positivity solutions.}$$

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Some examples of Diophantine equations similarly using mutations from cluster theory

Markov

$$M(x_1, x_2, x_3) := x_1^2 + x_2^2 + x_3^2 - 3x_1x_2x_3 = 0$$

Hone & Swart

$$H(x_1, x_2, x_3, x_4) := x_1^2x_4^2 + x_1x_3^3 + x_2^3x_4 + x_2^2x_3^2 - 4x_1x_2x_3x_4 = 0$$

Lampe

$$L(x_1, x_2, x_3, x_4, x_5) :=$$

$$x_1x_2(x_3^2 + x_4^2 + x_5^2) + (x_1^2 + x_2^2 + x_3x_4)(x_3 + x_4)x_5 - 9x_1x_2x_3x_4 = 0$$

Gyoda & Matsushita

$$G(x_1, x_2, x_3) :=$$
$$x_1^2 + x_2^4 + x_3^4 + 2x_1x_2^2 + kx_2^2x_3^2 + 2x_1x_3^2 - (7 + k)x_1x_2^2x_3^2 = 0$$

What is their common characteristc? “Almost Symmetry”?

Def: 1-cluster symmetric map

- Given $\sigma \in \mathfrak{S}_n$, $s \in [1,n]$, an integer vector $\mathbf{b} \in \mathbb{Z}^n$ with $b_s = 0$, call $(\sigma, s, \mathbf{b})$ a **seedlet.**
- **1-cluster symmetric map of $(\sigma, s, \mathbf{b})$** is defined as $\psi_{\sigma,s,\mathbf{b}}(\mathbf{x}) :=$

$$\left(x_{\sigma(1)}, \cdots, x_{\sigma(t-1)}, \frac{\prod_{j\in[1,n]} x_j^{[b_j]_+} + \prod_{j\in[1,n]} x_j^{[-b_j]_+}}{x_s}, x_{\sigma(t+1)}, \cdots, x_{\sigma(n)}\right),$$

  where $t = \sigma^{-1}(s)$.
- Briefly,

$$\psi_{\sigma,s,\mathbf{b}}(\mathbf{x}) = \Big(\sigma(\mathbf{x})\Big)\Big|_{\frac{\prod_{j\in[1,n]} x_j^{[b_j]_+} + \prod_{j\in[1,n]} x_j^{[-b_j]_+}}{x_s} \leftarrow x_s}.$$

- $\psi_{\sigma,s,\mathbf{b}} = \sigma\mu_s = \mu_{\sigma^{-1}(s)}\sigma$.

1-cluster symmetric group

Let $\Omega = (B, \mathbf{x})$ be a seed. For any permutation $\sigma \in \mathfrak{S}_n$.
The **permutation** σ of the seed Ω is defined to be the new seed $\sigma(B, \mathbf{x}) := (B', \mathbf{x}')$ given by

$$b'_{ij} = b_{\sigma(i)\sigma(j)}, \quad x'_i = x_{\sigma(i)}.$$

Proposition 3

(Bao-L.)
For a mutation μ_s, if $\sigma\mu_s(B, \mathbf{x}) = (\pm B, \mathbf{x}')$, then $\sigma\mu_s$, treating as transformation of variables, is the 1-cluster symmetric map of $(\sigma, s, \mathbf{b}_s)$, that is

$$\sigma\mu_s = \psi_{\sigma,s,\mathbf{b}_s}.$$

The 1-cluster symmetric group of the seed Ω *defined as*

$$\mathcal{G}_1(\Omega) := \langle \sigma\mu_s \mid \sigma\mu_s(B, \mathbf{x}) = (\pm B, \mathbf{x}'), \forall s \in [1, n], \sigma \in S_n \rangle$$

From given cluster algebra to find polynomials as invariants

How to characterize a Laurent polynomial which is invariant under a given 1-cluster symmetric map?

Theorem 4 (Bao-L.)

For $R = \mathbb{Z}$ or $= F_q$, given a 1-cluster symmetric map $\psi_{\sigma,s,\mathbf{b}}$. Let $F(\mathbf{x})$ be a Laurent polynomial in $R[\mathbf{x}^{\pm}]$ and its expression is

$$F(\mathbf{x}) = \mathbf{x}^{-\mathbf{d}} \sum_{\mathbf{j}\in\mathcal{N}} a_{\mathbf{j}}\mathbf{x}^{\mathbf{j}}.$$

with $\eta \in \mathbb{N}^{\mathbf{n}}, \mathbf{d} \in \mathbb{Z}^{\mathbf{n}}, \mathbf{d} = \sigma(\mathbf{d}), \eta_{\mathbf{s}} = \eta_{\mathbf{t}} = \mathbf{2d_s} = \mathbf{2d_t}$. (*)
Then the relation

$$F(\psi_{\sigma,s,\mathbf{b}}(\mathbf{x})) = F(\mathbf{x}) \quad (**)$$

holds, if and only if, the coefficients $\{a_{\mathbf{j}} \in R \mid \mathbf{j} \in \mathcal{N}\}$ satisfy the systems of homogeneous linear equations $HLE(\sigma, s, \mathbf{b}, \eta, \mathbf{d}, k)$ and $HLE(\sigma^{-1}, t, \mathbf{b}, \eta, \mathbf{d}, k)$.

$HLE(\sigma, s, \mathbf{b}, \eta, \mathbf{d}, k)$:

$$\begin{cases} 0 = a_{\sigma(\mathbf{j})} - \sum\limits_{\substack{0 \le l \le k \\ \mathbf{j} - \mathbf{b}^{(2k)}_{s,k,l} \in \mathcal{N}}} a_{\mathbf{j} - \mathbf{b}^{(2k)}_{s,k,l}} C^l_k, & \text{if } \mathbf{j} \in A, \\ 0 = \sum\limits_{\substack{0 \le l \le k \\ \mathbf{j} - \mathbf{b}^{(2k)}_{s,k,l} \in \mathcal{N}}} a_{\mathbf{j} - \mathbf{b}^{(2k)}_{s,k,l}} C^l_k, & \text{if } \mathbf{j} \in B, \\ 0 = a_{\sigma(\mathbf{j})}, & \text{if } \mathbf{j} \in C, \end{cases}$$

where $A = \pi_s^{(d_s - k)}\left(\sigma^{-1}(\mathcal{N}) \cap \bigcup_{0 \le l \le k} (\mathcal{N} + \mathbf{b}^{(2k)}_{s,k,l})\right)$,

$B = \pi_s^{(d_s - k)}\left(\bigcup_{0 \le l \le k} (\mathcal{N} + \mathbf{b}^{(2k)}_{s,k,l}) \setminus \sigma^{-1}(\mathcal{N})\right)$,

$C = \pi_s^{(d_s - k)}\left(\sigma^{-1}(\mathcal{N}) \setminus \bigcup_{0 \le l \le k} (\mathcal{N} + \mathbf{b}^{(2k)}_{s,k,l})\right)$.

$HLE(\sigma^{-1}, t, \mathbf{b}, \eta, \mathbf{d}, k)$:

$$\begin{cases} 0 = a_{\sigma^{-1}(\mathbf{j})} - \sum\limits_{\substack{0 \le l \le k \\ \mathbf{j} - \mathbf{v}_{t,k,l}^{(2k)} \in \mathcal{N}}} a_{\mathbf{j} - \mathbf{v}_{t,k,l}^{(2k)}} C_k^l, & \text{if } \mathbf{j} \in A', \\ 0 = \sum\limits_{\substack{0 \le l \le k \\ \mathbf{j} - \mathbf{v}_{t,k,l}^{(2k)} \in \mathcal{N}}} a_{\mathbf{j} - \mathbf{v}_{t,k,l}^{(2k)}} C_k^l, & \text{if } \mathbf{j} \in B', \\ 0 = a_{\sigma^{-1}(\mathbf{j})}, & \text{if } \mathbf{j} \in C', \end{cases}$$

where $A' = \pi_t^{(d_t - k)}\left(\sigma(\mathcal{N}) \cap \bigcup_{0 \le l \le k}(\mathcal{N} + \mathbf{v}_{t,k,l}^{(2k)})\right)$,

$B' = \pi_t^{(d_t - k)}\left(\bigcup_{0 \le l \le k}(\mathcal{N} + \mathbf{v}_{t,k,l}^{(2k)}) \setminus \sigma(\mathcal{N})\right)$,

$C' = \pi_t^{(d_t - k)}\left(\sigma(\mathcal{N}) \setminus \bigcup_{0 \le l \le k}(\mathcal{N} + \mathbf{v}_{t,k,l}^{(2k)})\right)$.

and where $\mathcal{N} := \{\mathbf{j} \in \mathbb{Z}^n_{\geq 0} \mid 0 \leq \pi_i(\mathbf{j}) \leq \pi_i(\eta), \forall\, i \in [1, n]\}$
and $\pi_s^{(k)}(\mathcal{N}) := \{\mathbf{j} \in \mathcal{N} \mid \pi_s(\mathbf{j}) = k\}$,

$$\mathbf{b}^{(i)}_{s,k,l} := l[\mathbf{b}]_+ + (k-l)[-\mathbf{b}]_+ - i\mathbf{e}_s,$$

where for $k = 0$, we say $C^l_0 = 1$.

Laurent polynomials from these equations

From Markov to build:

$$F_M(x_1, x_2, x_3) := \frac{x_1^2+x_2^2+x_3^2}{x_1 x_2 x_3} - 3$$

From Hone & Swart to build:

$$F_H(x_1, x_2, x_3, x_4) := \frac{x_1^2 x_4^2 + x_1 x_3^3 + x_2^3 x_4 + x_2^2 x_3^2}{x_1 x_2 x_3 x_4} - 4$$

From Lampe to build:

$$F_L(x_1, x_2, x_3, x_4, x_5) := \frac{x_1 x_2 (x_3^2+x_4^2+x_5^2) + (x_1^2+x_2^2+x_3 x_4)(x_3+x_4) x_5}{x_1 x_2 x_3 x_4} - 9$$

From Gyoda & Matsushita to build:

$$F_G(x_1, x_2, x_3) := \frac{x_1^2+x_2^4+x_3^4+2x_1 x_2^2+k x_2^2 x_3^2+2x_1 x_3^2}{x_1 x_2^2 x_3^2} - (7+k)$$

The operation is:

(1): Attempt to construct a Laurent polynomial from the original Diophantine equation such that the condition $(*)$ holds,

(2): Verify whether the equations $HLE(\sigma, s, \mathbf{b}, \eta, \mathbf{d}, k)$ and $HLE(\sigma^{-1}, t, \mathbf{b}, \eta, \mathbf{d}, k)$ hold,

(3): If not, adjust the approach in **(1)** and re-check **(2)**.

Invariant

Then, the Laurent polynomial is invariant under 1-cluster symmetric map, that is,

Markov:

$$F_M(\mu_i(x_1, x_2, x_3)) = F_M(x_1, x_2, x_3), \ i = 1, 2, 3$$

Hone & Swart:

$$F_H(\sigma_{(1234)}\mu_1(x_1, x_2, x_3, x_4)) = F_H(x_1, x_2, x_3, x_4)$$

Lampe

$$F_L(\sigma_{(12)}\mu_1(x_1, x_2, x_3, x_4, x_5)) = F_L(x_1, x_2, x_3, x_4, x_5)$$

It follows that the positivity of cluster variables in $\mathbb{Z}$-cluster algebras is used in the operation from an initial solution $\mathbf{x}_0 \in \mathbb{N}^n$.

Laurent polynomial as invariant and the orbit of positive integer solutions

Proposition 5

(Bao-L.)
For a seed Ω *of a* $\mathbb{Z}$*-cluster algebra* $\mathcal{A}(\Omega)$*, then* $\mathcal{G}_1(\Omega)(\mathbf{1}) \subset \mathbb{Z}^n_{>0}$
Suppose $F(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}^{\pm}]^{\mathcal{G}_1(\Omega)}$*. Then*

$$\mathcal{G}_1(\Omega)(\mathbf{1}) \subset \mathcal{V}_{\mathbb{Z}_{>0}}(F(\mathbf{x}) - F(\mathbf{1})).$$

is an orbit.

Basic idea

Why do we need cluster algebras over finite fields?

As shown above, certain specific Diophantine equations are linked to $\mathbb{Z}$-cluster algebras, since the solutions of the former can be classified into orbits via mutation mappings of the latter.

In number theory, the local-global principle is important. For example, the relation between positive integer solutions of Diophantine equations and their solutions over finite fields. Under this view, F_q-cluster algebras becomes necessary, since it allows us to connect the solutions of specific equations over finite fields with the mutation mappings of F_q-cluster algebras to obtain classification of orbits.

Then, we can establish a “global-local” relation between the orbit classification of solutions to Diophantine equations and that to the corresponding equations over finite fields.

A canonical map from **Z**-equations to F_p-equations

For a prime p, we have the canonical map

$$\pi : \mathbb{Z} \to F_p = \mathbb{Z}/p\mathbb{Z}, \text{ via } z \mapsto \bar{z}.$$

So, for a given Diophantine equation satisfying 1-cluster symmetry: $f(\mathbf{x}) = \sum_{(i_1,i_2,\cdots,i_n)\in\mathbb{N}^n} a_{i_1 i_2 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} = 0$ where all $a_{i_1 i_2 \cdots i_n} \in \mathbb{Z}$, with the 1-cluster symmetric group $\mathcal{G}_1(\Omega)$,

let $\mathcal{V}_{\mathbb{Z}>0}$ be the set of positive integer solutions of $f(\mathbf{x}) = 0$. Then the group $\mathcal{G}_1(\Omega)$ acts on $\mathcal{V}_{\mathbb{Z}>0}$.

We have the corresponding polynomial equation over F_p satisfying 1-cluster symmetry:

$$\bar{f}(\mathbf{x}) = \sum_{(i_1,i_2,\cdots,i_n)\in\mathbb{N}^n} \bar{a}_{i_1 i_2 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} = 0$$

where all $\bar{a}_{i_1 i_2 \cdots i_n} \in \mathbb{F}_p$, with the same 1-cluster symmetric group $\mathcal{G}_1(\Omega)$,

let $\mathcal{V}_{F_p}$ be the set of solutions of $f(\mathbf{x}) = 0$ over F_p. Then the group $\mathcal{G}_1(\Omega)$ acts on $\mathcal{V}_{F_p}$.

Pre-image of F_p-solution in $\mathcal{V}_{\mathbb{Z}>0}$

From the canonical map: $\pi : \mathbb{Z} \to F_p = \mathbb{Z}/p\mathbb{Z}$, via $z \mapsto \bar{z}$.
We induce the map: $\hat{\pi} : \mathcal{V}_{\mathbb{Z}>0} \to \mathcal{V}_{F_p}$, via $\mathbf{x}_0 \mapsto \hat{\mathbf{x}}_0$.
We give a characterization for $\hat{\pi}$ to be **surjective** as follows.

Proposition 6 (By Dekker for Markov equation)

Let p be prime. Then every solution to the Markoff equation over $\mathbb{F}_p$ has some pre-image in $\mathbb{N}^3$ if and only if $\mathcal{V}_{F_p}$ is a connected graph under action of 1-cluster symmetric group.

Proposition 7 (By L.-Pan for general equations)

Assume the set of positive integer solutions $\mathcal{V}_{\mathbb{Z}>0}$ of a 1-cluster symmetric Diophantine equation $f(\mathbf{x}) = 0$ has ***only one*** *orbit under the action of $\mathcal{G}_1(\Omega)$ (or say, is connected).*
Then every solution of $\bar{f}(\mathbf{x}) = 0$ has some pre-image in $\mathcal{V}_{\mathbb{Z}>0}$ if and only if the solution set $\mathcal{V}_{F_p}$ of $\bar{f}(\mathbf{x}) = 0$ is connected.

Strong approximation conjecture by A.Baragar

Conjecture 1

For any prime p, the non-zero solutions set $\mathcal{V}_{F_p}$ of Markov equation $x^2 + y^2 + z^2 = 3xyz$ over F_p is connected.

[1] P Sarnak, etc, C. R. Math. Acad. Sci. Paris 354(2) (2016).

By Prop.6, $\hat{\pi} : \mathcal{V}_{\mathbb{Z}>0} \to \mathcal{V}_{F_p}$ is surjective if this conjecture hold.

The first major progress is the work in [2]:

[2] W Y. Chen, Ann. of Math. (2) 199 (2024), no. 1.

in which it was proved that the cardinality of a connected component of $\mathcal{G}_p$ is divisible by p and in particular,

the conjecture holds for all but finitely many primes p.

Pproposal: *For finitely many primes p_0 which are not been known if satisfying Conjecture 1, we may use Proposition 6, 7 to discuss if there is a pre-image in $\mathcal{V}_{\mathbb{Z}>0}$ for any solution in $\mathcal{V}_{F_p}$.*

Examples with unique orbit of solution set over $\mathbb{Z}_{>0}$

$$\Omega_1, \qquad F_1 : \frac{x^2+y^2+z^2}{xyz}$$

$$\Omega_2, \qquad F_2 : \frac{x^2+y^2+z^2+k_1yz+k_2zx+k_3xy}{xyz}$$

$$\Omega_3, \qquad F_3 : \frac{x^2+y^4+z^4+2xy^2+ky^2z^2+2xz^2}{xy^2z^2}$$

$$\Omega_4, \qquad F_4 : \frac{x^2+y^4+z^4+2x(y^2+z^2)+k_1yz(x+y^2+z^2)+k_2y^2z^2}{xy^2z^2}$$

Then, $\boxed{\mathcal{G}_1(\Omega_i)(\mathbf{1}) = \mathcal{V}_{\mathbb{Z}_{>0}}(F_i(\mathbf{x}) - F_i(\mathbf{1}))}$ for $i = 1, 2, 3, 4$. where all Ω_i are the corresponding seeds from some cluster algebras.

In particular, the result on the equation F_4 is given by us.

Example for non-unique orbits by Lampe[2]

- Given a seed $\Omega := (B, \mathbf{x})$,

$$B := \begin{pmatrix} 0 & -2 & 1 & 1 & 0 \\ 2 & 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & 1 & -1 \\ -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

- Clearly, $\sigma_{(12)}\mu_1, \sigma_{(1234)}\mu_4 \in \mathcal{G}_1(\Omega)$. Denote a group

$$G := \langle \sigma_{(12)}\mu_1, \sigma_{(1234)}\mu_4 \rangle.$$

- Then $L(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}^{\pm}]^G$, where

$$L(\mathbf{x}) := \frac{x_1 x_2(x_3^2 + x_4^2 + x_5^2) + (x_1^2 + x_2^2 + x_3 x_4)(x_3 + x_4)x_5}{x_1 x_2 x_3 x_4} - 9$$

- Fix $k \in \mathbb{Z}_{>0}$, then $L(k\mathbf{1}) = 0$ and $G(k\mathbf{1}) \subsetneq \mathcal{V}_{\mathbb{Z}_{>0}}(L)$.

[2] P. Lampe. Diophantine equations via cluster transformations. JA (2016)

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$$G := \langle \sigma_{(12)}\mu_1, \sigma_{(1234)}\mu_4 \rangle.$$

- Then $L(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}^{\pm}]^G$, where

$$L(\mathbf{x}) := \frac{x_1 x_2 (x_3^2 + x_4^2 + x_5^2) + (x_1^2 + x_2^2 + x_3 x_4)(x_3 + x_4) x_5}{x_1 x_2 x_3 x_4} - 9$$

- Fix $k \in \mathbb{Z}_{>0}$, then $L(k\mathbf{1}) = 0$ and $G(k\mathbf{1}) \subsetneq \mathcal{V}_{\mathbb{Z}_{>0}}(L)$.

[2] P. Lampe. Diophantine equations via cluster transformations. JA (2016)

Example for non-unique orbits by Lampe[2]

- Given a seed $\Omega := (B, \mathbf{x})$,

$$B := \begin{pmatrix} 0 & -2 & 1 & 1 & 0 \\ 2 & 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & 1 & -1 \\ -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

- Clearly, $\sigma_{(12)}\mu_1, \sigma_{(1234)}\mu_4 \in \mathcal{G}_1(\Omega)$. Denote a group

$$G := \langle \sigma_{(12)}\mu_1, \sigma_{(1234)}\mu_4 \rangle.$$

- Then $L(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}^{\pm}]^G$, where

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Characterization for $G(k\mathbf{1})$

Theorem 8 (Bao-L.)

$$G(k\mathbf{1}) = \mathcal{V}_{\mathbb{Z}_{>0}}(L) \cap \{\mathbf{x}' \in \mathbb{Z}^5_{>0} \mid \varphi(\mathbf{x}') \in \mathbb{Z}^3_{>0}, \mathbf{x}' \equiv 0(mod\ k)\}.$$

where $\mathbf{x}' \equiv 0 (\mathrm{mod}\ k)$ means $x_i \equiv 0 (\mathrm{mod}\ k)$ for all $i \in [1, n]$ and $\varphi : (a, b, c, d, e) \mapsto$

$$\left(\frac{a^2 + b^2 + cd}{ab}, \frac{c^2 d + a^2 c + b^2 d + abe}{bcd}, \frac{cd^2 + a^2 c + b^2 d + abe}{acd}\right)$$

[3] L. Bao & F. Li, A study on Diophantine equations via cluster theory, J. Algebra 639:99 – 119, 2024.

On the equation $H: \ x^2 + y^2 + 1 = 3xy$

♠ This equation is the special case of Markov equation for $z = 1$, that is, its solution graph can be embedded into the Markov graph.

♠ This equation is a 1-cluster symmetric equation with exchange matrix: $B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ and the mutations of the corresponding cluster algebra.

♠ The other meaning of this equation is that it can be written as a quadratic equation $x^2 + y^2 - 3xy = -1$, so it has rational solutions over $\mathbb{Q}$ if and only if it has p-adic solutions over the p-adic field $\mathbb{Q}_p$ by Hasse Theorem due to the local-global principle.

Connection of solutions of Equation H over $\mathbb{Z}_{>0}$

All $\mathbb{Z}_{>0}$-solutions of Equation H can be obtained from initial solution $(1,1)$ through finite many mutations μ_i defined satisfying:

$$\mu_1(x,y) = (\frac{y^2+1}{x}, y), \mu_2(x,y) = (x, \frac{x^2+1}{y})$$

The 1-cluster symmetric group $\mathcal{G}_1$ is generated as $\langle \mu_1, \mu_2 \rangle$ with relations $\mu_1^2 = 1, \mu_2^2 = 1$.

Then the solution set $\mathcal{V}_{\mathbb{Z}_{>0}}(F)$ of Equation H is just the unique orbit under action of the group $\mathcal{G}_1$, that is,

$$\mathcal{G}_1(1,1) = \mathcal{V}_{\mathbb{Z}_{>0}}(F)$$

The solution set of Equation H over F_p

For the equation H over F_p, we can replace $\frac{y^2+1}{x}$, $\frac{x^2+1}{y}$ with $3y - x,\ 3x - y$ respectively, then we have

$$\mu_1(x,y) = (3y - x, y),\ \mu_2(x,y) = (x, 3x - y) \tag{3}$$

such that we can perform mutation along some direction $k = 1, 2$ even if the coordinate of this direction of the solution over F_p are zero.

We call such solution an **ambiguous solution**.

Is the solution set of Equation H connected over F_p?

We have known the solution set of Markov equation over F_p has been proved to be connected for all p but finitely many primes as the answer for Conjecture 1.

Now we consider the same problem for the equation H, that is,

Is the solution set of the equation H over F_p connected as graph under mutations?

The examples we will give below show for some p the solution set of H is connected and for other p that is not connected.

1-cluster symmetric group and orbit over F_5 (1)

Over F_5, the solution set $\mathcal{V}_{F_5}(H)$ of $H: x^2 + y^2 + 1 = 3xy$ contains 10 various solutions:

$(1,1),(1,2),(2,1),(0,2),(0,3),(2,0),(3,0),(4,4),(4,3),(3,4),$

where $(0,2),(2,0),(3,0),(0,3)$ are ambiguous solutions.

In this case we have the replacing mutations in (3), that is,

$$\mu_1(x,y) = (3y - x, y), \quad \mu_2(x,y) = (x, 3x - y)$$

1-cluster symmetric group and orbit over F_5 (2)

Under the action of the 1-cluster symmetric group $\mathcal{G}_{\mathbb{F}_5} = \langle \mu_1, \mu_2 \rangle$, the solution set $\mathcal{V}_{\mathbb{F}_5}(H)$ of Equation H forms the following connected graph:

$$
\begin{array}{ccccccccc}
(2,1) & \overset{2}{\text{———}} & (2,0) & \overset{1}{\text{———}} & (3,0) & \overset{2}{\text{———}} & (3,4) & \overset{1}{\text{———}} & (4,4) \\
\Big| {\scriptstyle 1} & & & & & & & & \Big| {\scriptstyle 2} \\
(1,1) & \overset{2}{\text{———}} & (1,2) & \overset{1}{\text{———}} & (0,2) & \overset{2}{\text{———}} & (0,3) & \overset{1}{\text{———}} & (4,3)
\end{array}
$$

That is, the orbit is unique.

1-cluster symmetric group and orbit over F_5 (3)

In addition to the relations $\mu_1^2 = 1$ and $\mu_2^2 = 1$, the 1-cluster symmetric group $\mathcal{G}_{\mathbb{F}_5}$ also has the following new relations:

$$(\mu_1\mu_2)^5 = 1, \quad (\mu_2\mu_1)^5 = 1$$

Based on this, we obtain the following characterization:

Proposition 9

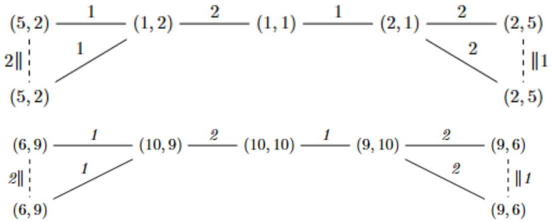
The 1-cluster symmetric group $\mathcal{G}_{\mathbb{F}_5}$ of the equation $H : x^2 + y^2 + 1 = 3xy$ over the field $\mathbb{F}_5$ is isomorphic to the dihedral group D_5.

1-cluster symmetric group and orbit over F_{11} (1)

Over the finite field $\mathbb{F}_{11}$, the solution set $\mathcal{V}_{\mathbb{F}_{11}}(H)$ of the equation $x^2 + y^2 + 1 = 3xy$ consists of exactly 10 solutions:

$$(1,1),(1,2),(2,1),(5,2),(2,5),(10,10),(10,9),(9,10),(9,6),(6,9)$$

Under the action of the 1-cluster symmetric group $\mathcal{G}_{\mathbb{F}_{11}} = \langle \mu_1, \mu_2 \rangle$, the solution set $\mathcal{V}_{\mathbb{F}_{11}}(H)$ of equation H forms the following two connected graphs:



That is, in this case, the solution set $\mathcal{V}_{\mathbb{F}_{11}}(H)$ contains two orbits.

1-cluster symmetric group and orbit over F_{11} (2)

For each orbit, there is a new relation:

$$(\mu_2\mu_1)^5 = 1, \quad (\mu_1\mu_2)^5 = 1$$

Based on this, we obtain the following characterization:

Proposition 10

The 1-cluster symmetric group $\mathcal{G}_{\mathcal{T}_\nabla}$ of the equation $H : x^2 + y^2 + 1 = 3xy$ over the field $\mathbb{F}_{11}$ is isomorphic to $D_5 \times D_5$.

A result on pre-images of orbits

Note that the solution set of Equation H has two orbits over $\mathbb{F}_{11}$, it is easy to see in the second orbit any solution has no preimage in $\mathbb{N}^2$.

In fact, we have the following general result:

Theorem 0.0.1

For a 1-cluster symmetric F_p-polynomial $f(X) = 0$ for any prime p, all solutions in its one orbit of the solution set **at the same time either have pre-images or have not pre-images in $\mathbb{N}^n$.**

Thanks !