Quantization of minimal nilpotent orbits and the quantum Hikita conjecture

Joint with Weiqiang He and Sirui Yu

arXiv: 2302.13249

Xiaojun Chen
New Uzbekistan University & Sichuan University

January 19, 2026

A motivating example

Recall that $\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} w & u \\ v & -w \end{pmatrix} \middle| u, v, w \in \mathbb{C} \right\}$. Inside $\mathfrak{sl}_2(\mathbb{C})$, an interesting geometric object is the set of nilpotent elements (called the nilpotent cone)

$$\mathcal{N} := \left\{ \begin{pmatrix} w & u \\ v & -w \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}) \middle| w^2 + uv = 0 \right\} \cong \mathbb{C}^2/\mathbb{Z}_2,$$

which is also called the A_1-singularity.

In algebraic geometry, $\mathbb{C}^2/\mathbb{Z}_2$ has a minimal resolution

$$\widetilde{\mathcal{N}} \to \mathcal{N},$$

where $\widetilde{\mathcal{N}}$ is given by

$$\left\{(A, L) \in \mathcal{N} \times \mathbb{P}^1 : L \subset \mathbb{C}^2, A \cdot \mathbb{C}^2 \subset L, AL = 0\right\}.$$

It is easy to see that $\widetilde{\mathcal{N}} \cong T^*\mathbb{P}^1$. Since $T^*\mathbb{P}^1$ is homotopic to $\mathbb{P}^1$, we have

Fact 1

The cohomology ring $\mathrm{H}^\bullet(\widetilde{\mathcal{N}}, \mathbb{C})$ is isomorphic to $\mathbb{C}[x]/x^2$.

Now consider the Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$:

$$\mathfrak{h} := \left\{ \begin{pmatrix} w & 0 \\ 0 & -w \end{pmatrix} \right\} \in \mathfrak{sl}_2(\mathbb{C}).$$

Formally, its intersection with $\mathcal{N} = \left\{ \begin{pmatrix} w & u \\ v & -w \end{pmatrix} \middle| w^2 + uv = 0 \right\}$ is

$$\mathfrak{h} \cap \mathcal{N} = \left\{ \begin{pmatrix} w & 0 \\ 0 & -w \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}) \middle| w^2 = 0 \right\}.$$

Fact 2

The "scheme theoretic intersection" of $\mathcal{N}$ with $\mathfrak{h}$, or in other words, the $\mathbb{C}^*$-fixed points of $\mathcal{N}$, is

$$\mathcal{O}(\mathcal{N}^{\mathbb{C}^*}) = \mathbb{C}[x]/x^2.$$

In summary, we have the following isomorphism: Let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{sl}_2(\mathbb{C})$. Denote by $\widetilde{\mathcal{N}}$ the minimal resolution of $\mathcal{N}$. Then we have isomorphism

$$\mathrm{H}^\bullet(\widetilde{\mathcal{N}}, \mathbb{C}) \cong \mathcal{O}(\mathcal{N}^{\mathbb{C}^*}).$$

A natural question is:

Question

Does the above isomorphism hold by coincidence, or may there be a theory behind it?

Answer by Hikita: this isomorphism is a property of *symplectic duality*, or equivalently, a property of *3d* $\mathcal{N} = 4$ *mirror symmetry*.

What is symplectic duality?

Definition (Braden, Licata, Proudfoot, Webster)

Let X and $X^!$ be two (possibly singular) symplectic varieties. They are said to be *symplectic dual* to each other if they satisfy several (totally seven) highly nontrivial identities.

These identities involve many recent results in algebra, (complex and algebraic) geometry, (algebraic and geometric) representation theory and even mathematical physics. For example, the authors generalized the concept "category O" in representation theory to symplectic varieties, and propose that symplectic dual pairs of varieties should have their category O's Koszul dual to each other.

A good reference for this topic is Kamnitzer's survey paper: Kamnitzer, *Symplectic resolutions, symplectic duality, and Coulomb branches*, Bulletin of the London Mathematical Society 54 (2022), no. 5, 1515-1551. The following picture is taken from the paper:

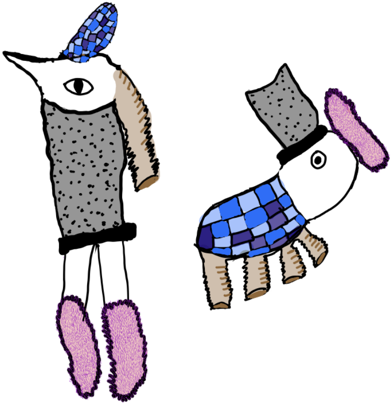

FIGURE 3. An artist's view of symplectic duality: certain structures on one creature match different structures on the other creature.

Example

1. The nilpotent cone $\mathcal{N}$ in $\mathfrak{sl}_2(\mathbb{C})$, with dual being itself, satisfies all the axioms of symplectic duality. Hence $\mathcal{N}$ is symplectic dual to itself.
2. Hilbert schemes of points $\mathrm{Hilb}^n(\mathbb{C}^2)$ in $\mathbb{C}^2$ are symplectic self-dual.
3. Let G and ${}^LG$ be two Langlands dual Lie groups. Denote by B and ${}^LB$ the Borel subgroups of G and ${}^LG$ respectively. Then T^*G/B and $T^*({}^LG/{}^LB)$ are symplectic dual.

Hikita's conjecture

One of the axioms in symplectic duality is the following:

Conjecture (Hikita)

Let X and $X^!$ be a pair of symplectic dual *conical symplectic varieties* over $\mathbb{C}$. Suppose $X^!$ admits a conical symplectic resolution $\widetilde{X}^! \to X^!$, and T is a maximal torus of the Hamiltonian action on X. Then there is an isomorphism of graded algebras

$$\mathrm{H}^\bullet(\widetilde{X}^!, \mathbb{C}) \cong \mathbb{C}[X^T].$$

The goal of this talk is to generalize our previous example of the nilpotent cone in $\mathfrak{sl}_2(\mathbb{C})$ to the case of other types of Lie algebras, with a focus on the (quantum version of) Hikita conjecture.

The $\mathfrak{sl}_{n+1}(\mathbb{C})$ case

Let $\mathcal{N}$ be the set of nilpotent elements in $\mathfrak{sl}_{n+1}(\mathbb{C})$. It is the disjoint union of $\mathrm{SL}_{n+1}(\mathbb{C})$-orbits, called the *nilpotent orbits*. They are parameterized by the Jordan forms of the nilpotent elements, and hence are also in one-to-one correspondence with the set of partitions of $n+1$.

Object 1: Minimal nilpotent orbit

Our first object is

$$\overline{O}_{min} := \{A \in \mathcal{N} : \mathrm{rk}(A) \leq 1\},$$

which is the closure of the *minimal nilpotent orbit*. Elements in $\overline{O}_{min}$ are either 0 or having the Jordan form

$$\begin{pmatrix} 0 & 1 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Like before, $\overline{O}_{min}$ has a crepant resolution $T^*\mathbb{P}^n$.

Object 2: Slodowy variety

Our second object is given as follows: suppose $x \in \mathcal{N}$ has rank $n-1$. In this case, x is called a *sub-regular* nilpotent element. By the Jacobson-Morozov theorem, there are $y, h \in \mathfrak{sl}_{n+1}(\mathbb{C})$, such that x, y, h form an $\mathfrak{sl}_2$ sub-algebra, called the $\mathfrak{sl}_2$-tripple. Let $S_x = x + \ker[y, -] \subset \mathfrak{sl}_{n+1}(\mathbb{C})$, which is usually called the *Slodowy slice*. Then we have the following

Theorem (Brieskorn-Slodowy)

$$S_x \cap \mathcal{N} \cong \mathbb{C}^2/\mathbb{Z}_{n+1} \qquad (\textit{the } A_n\textit{-singularity})$$

In algebraic geometry, the A_n-singularity has a minimal resolution, which we denote by $\widetilde{S_x \cap \mathcal{N}}$. In literature, $\widetilde{S_x \cap \mathcal{N}}$ is usually called the *Slodowy variety*.

We may also apply similar constructions to D and E type Lie algebras, and obtain the corresponding minimal nilpotent orbits and Slodowy varieties.

Theorem (Nakajima, Kamnitzer, Krylov, ...)

Let $\mathfrak{g}$ be a Lie algebra of type A, D or E. Then $\overline{O}_{min}$ and $S_x \cap \mathcal{N}$ are symplectic dual to each other. In particular, Hikita's conjecture hods (due to Krylov):

$$\mathrm{H}^\bullet(\widetilde{S_x \cap \mathcal{N}}) \cong \mathcal{O}(\overline{O}_{min}^{\mathbb{C}^*}).$$

In the above theorem, $S_x \cap \mathcal{N}$ is isomorphic to singularity of type A, D or E, depending on the type of $\mathfrak{g}$. In particular, if $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, it becomes the isomorphism in our starting example.

What next?

Question

1. Since the Slodowy varieties and the minimal nilpotent orbit closure both admit a $\mathbb{C}^*$-action, do we have an *equivariant* version of the above isomorphism? This is called the Hikita-Nakajima conjecture. Or more generally, do we have a *quantum equivariant* version of the above isomorphism (due to Kamnitzer and his collaborators)?
2. Do we have similar results for other types of Lie algebras, that is, for Lie algebras of type B, C, F and G?

Equivariant cohomology of ADE singularities

Let X be an ADE singularity, constructed as before. X can also be realized as $\mathbb{C}^2/\Gamma$, where $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ is a finite group. For example, for A_n singularity, $\Gamma = \mathbb{Z}_{n+1}$, with $\Gamma \times \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$\left(e^{2\pi i/(n+1)}, (z_1, z_2)\right) \mapsto \left(e^{2\pi i/(n+1)} \cdot z_1, e^{-2\pi i/(n+1)} \cdot z_2\right).$$

There is a natural $\mathbb{C}^*$-action on A_n given by

$$(t, [z_1, z_2]) \mapsto [t \cdot z_1, t \cdot z_2].$$

This action extends to the minimal resolution $\widetilde{A}_n$, and thus we may find the $\mathbb{C}^*$-equivariant cohomology $\mathrm{H}^\bullet_{\mathbb{C}^*}(\widetilde{A}_n)$ of $\widetilde{A}_n$. It may be viewed as a "deformation" of $\mathrm{H}^\bullet(\widetilde{A}_n)$, since the generators remain the same, but the product of the generators, in the equivariant case, contains higher order terms.

Theorem (Bryan-Gholampour)

Let $\mathbb{C}^2/\Gamma$ *be an ADE singularity, and let* $\widetilde{\mathbb{C}^2/\Gamma}$ *be its minimal resolution. Then* $\mathrm{H}^\bullet_{\mathbb{C}^*}(\widetilde{\mathbb{C}^2/\Gamma})$ *has basis* $e_1, e_2, \cdots, e_n$ *(the same number of the vertices in the Dynkin diagram of the Lie algebra). For* $e_\alpha, e_{\alpha'} \in \mathrm{H}^\bullet_{\mathbb{C}^*}(\widetilde{\mathbb{C}^2/\Gamma})$*, the product is given by*

$$e_\alpha \star e_{\alpha'} = -t^2|\Gamma|\langle\alpha, \alpha'\rangle + \sum_{\gamma\in\Delta^+} t\langle\alpha, \gamma\rangle\langle\alpha', \gamma\rangle e_\gamma, \tag{1}$$

where $e_\alpha = c_1 e_1 + \cdots + c_n e_n$*, if* $\alpha = c_1\alpha_1 + \cdots c_n\alpha_n$*,* $c_1, \cdots, c_n \in \mathbb{N}$*.*

B-algebra of minimal nilpotent orbit closure

Definition (Drinfeld, Nakajima...)

Suppose $X = \operatorname{Spec} A$ is an affine scheme with a $\mathbb{C}^*$-action. Then the coordinate ring of the $\mathbb{C}^*$-fixed scheme is called the *B-algebra of A*.

Remark

This definition can be generalized to the "quantum version" (due to Nakajima?).

For $\mathfrak{g}$ a Lie algebra of ADE type, if we identity $\mathfrak{g}$ with $\mathfrak{g}^*$, then we may view $U(\mathfrak{g})$ as a quantization of $\mathfrak{g}$. Joseph and later Garfinkle constructed a two sided ideal J of $U(\mathfrak{g})$, called the *Joseph ideal*, such that $U(\mathfrak{g})/J$ is a quantization of $\overline{O}_{min}$. Denote the associated B-algebra by $B(\overline{O}_{min})$.

Main theorem 1

Theorem (C.-He-Yu)

Let $\mathfrak{g}$ be a Lie algebra of type A, D or E. Then the equivariant version of Hikita's conjecture (the Hikita-Nakajima conjecture) holds:

$$\mathrm{H}^{\bullet}_{\mathbb{C}^*}(\widetilde{S_x \cap \mathcal{N}}) \cong B(\overline{O}_{min}).$$

This gives an evidence that Kleinian singularities are symplectic dual to the minimal nilpotent orbit closure in the corresponding Lie algebra. (There are some more evidences due to Nakajima and others.)

The BCFG type Lie algebra case

For BCFG type Lie algebras, we have similar result. Before going to the details, let us remind that Lie algebras of type B and C are Langldands dual to each other, while the rest types of Lie algebras are Langlands self-dual.

Definition (Slodowy)

Let $\mathfrak{g}$ be a Lie algebra of type BCFG. Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone. Let $x \in \mathcal{N}$ be a sub-regular nilpotent element. Then $S_x \cap \mathcal{N}$ is called the *simple singularity with the same type of* $\mathfrak{g}$. Moreover, they are isomorphic to to the singularities of type ADE with an extra $\mathbb{Z}_2$- or $\mathfrak{S}_3$-action.

More precisely, Slodowy obtained the following

Type of singularity	ADE singularity	Corresp. group
B_n	A_{2n-1}	$\mathbb{Z}_2$
C_n	D_{n+1}	$\mathbb{Z}_2$
F_4	E_6	$\mathbb{Z}_2$
G_2	D_4	$\mathfrak{S}_3$

On the other side, we have the following.

Theorem (Brylinski-Kostant)

Let $\mathfrak{g}$ be a Lie algebra of type BCFG. Then there exists a nilpotent orbit (called the special minimal orbit and denoted by O_{sm}) in $\mathfrak{g}$ such that it is covered by the minimal nilpotent orbit closure in Lie algebra of type ADE with deck transformation $\mathbb{Z}_2$ or $\mathfrak{S}_3$.

More precisely, we have the following table.

Type of $\mathcal{O}_{ms}$	Covering	Deck transformation
$\mathcal{O}_{ms}(B_n)$	$\mathcal{O}_{min}(D_{n+1})$	$\mathbb{Z}_2$
$\mathcal{O}_{ms}(C_n)$	$\mathcal{O}_{min}(A_{2n-1})$	$\mathbb{Z}_2$
$\mathcal{O}_{ms}(F_4)$	$\mathcal{O}_{min}(E_6)$	$\mathbb{Z}_2$
$\mathcal{O}_{ms}(G_2)$	$\mathcal{O}_{min}(D_4)$	$\mathfrak{S}_3$

Main theorem 2

Theorem (C.-He-Yu)

Let $\mathcal{B}_n$, $\mathcal{C}_n$, $\mathcal{F}_4$ and $\mathcal{G}_2$ be the minimal resolutions of singularities of B_n, C_n, F_4 and G_2 respectively, and let $\tilde{\mathcal{O}}_{ms}(B_n)$, $\tilde{\mathcal{O}}_{ms}(C_n)$, $\tilde{\mathcal{O}}_{ms}(F_4)$ and $\tilde{\mathcal{O}}_{ms}(G_2)$ be (the normalizations of) the closures of the minimal special nilpotent orbits in Lie algebras of BCFG type respectively. Then

$$\begin{aligned} \mathrm{H}^\bullet_{\mathbb{Z}_2\times\mathbb{C}^*}(\mathcal{B}_n) &\cong B(\tilde{\mathcal{O}}_{ms}(C_n)), & \mathrm{H}^\bullet_{\mathbb{Z}_2\times\mathbb{C}^*}(\mathcal{C}_n) &\cong B(\tilde{\mathcal{O}}_{ms}(B_n)), \\ \mathrm{H}^\bullet_{\mathbb{Z}_2\times\mathbb{C}^*}(\mathcal{F}_4) &\cong B(\tilde{\mathcal{O}}_{ms}(F_4)), & \mathrm{H}^\bullet_{\mathfrak{S}_3\times\mathbb{C}^*}(\mathcal{G}_2) &\cong B(\tilde{\mathcal{O}}_{ms}(G_2)). \end{aligned}$$

Remark

There is also a quantum version of the above two theorems.

Thank you!