Derived representation schemes and derived Schur algebras

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Representation functor

Let k be an algebraically closed field.

Let Alg_k be the category of associative k-algebras.

Let CAlg_k be the category of associative and commutative k-algebras.

For $A \in \mathrm{Alg}_k$ and $n \in \mathbb{N}$, consider

$$\mathrm{Rep}_n(A) : \mathrm{CAlg}_k \to \mathrm{Sets},\ B \mapsto \mathrm{Hom}_{\mathrm{Alg}}(A, M_n(B)). \tag{1}$$

Theorem (Bergman - Cohn, 1974)

The functor $\mathrm{Rep}_n(A)$ *is representable by* $A_n \in \mathrm{CAlg}_k$

$$\mathrm{Hom}_{\mathrm{CAlg}}(A_n, B) \cong \mathrm{Hom}_{\mathrm{Alg}}(A, M_n(B))$$

If we take $B = k$, then

$$\mathrm{Spec}(A_n) := \mathrm{Hom}_{\mathrm{CAlg}}(A_n, k) \cong \mathrm{Hom}_{\mathrm{Alg}}(A, M_n(k)).$$

So the algebra A_n is the coordinate ring of the variety of n-dimensional representations of A.

Idea of Bergman-Cohn's theorem :

They considered the functor

$$\widetilde{\mathrm{Rep}}_n(A) : \mathrm{Alg}_k \to \mathrm{Sets},\ B \mapsto \mathrm{Hom}_{\mathrm{Alg}}(A, M_n(B)). \tag{2}$$

and showed it is representable by $\sqrt[n]{A} \in \mathrm{Alg}_k$

$$\mathrm{Hom}_{\mathrm{Alg}}(A, M_n(B)) \simeq \mathrm{Hom}_{\mathrm{Alg}}(\sqrt[n]{A}, B)\,.$$

Then $A_n = (\sqrt[n]{A})_{ab}$ (algebra abelianization).

Lemma

For an associative algebra A,

$$\sqrt[n]{A} = (M_n(k) * A)^{M_n(k)}$$

where the RHS is the centralizer of the image of $M_n(k)$ *in* $M_n(k) * A$.

Thus, $\sqrt[n]{A}$ is generated by elements

$$a_{ij} := \sum_{m=1}^{n} e_{mi} a e_{jm},$$

where $a \in A$ and e_{ij} are the elementary matrices in $M_n(k)$.

Example

Let $A = \mathbb{C}\langle x, y \rangle$. Then $\sqrt[n]{A}$ is generated by x_{ij} and $y_{ij}(i, j = 1, ..., n)$. In fact,

$$\sqrt[n]{A} = \mathbb{C}\langle x_{ij}, y_{ij} \rangle \quad \text{and} \quad A_n = \mathbb{C}[x_{ij}, y_{ij}] .$$

From representability of functors $\mathrm{Rep}_n(A)$ and $\widetilde{\mathrm{Rep}}_n(A)$, we obtain adjoint pairs of functors

$$\sqrt[n]{-}: \ \mathrm{Alg}_k \leftrightarrows \mathrm{Alg}_k \ : M_n(-)$$
$$(-)_n : \ \mathrm{Alg}_k \leftrightarrows \mathrm{CAlg}_k \ : M_n(-)\,.$$

Remark

If the category Alg_k were abelian or exact category, one could ask whether functors $\sqrt[n]{-}$ and $(-)_n$ are exact, and if not, what their left derived functors would be.

Y.Berest, G.Khachatryan, and A.Ramadoss, *Derived representation schemes and cyclic homology*, **Adv. Math.** 245 (2013) 625-689

We have defined the functor

$$\mathrm{Rep}_n : \mathrm{Alg}_k \to \mathrm{CAlg}_k, \quad A \mapsto A_n$$

How do we find/define $\mathrm{DRep}_n(A)$?

Consider categories: $(\mathrm{DGA}_k/\mathrm{CDGA}_k)$.

1. $\mathrm{Alg}_k \hookrightarrow \mathrm{DGA}_k$ and $\mathrm{CAlg}_k \hookrightarrow \mathrm{CDGA}_k$
2. Rep_n should be extended to a functor $\mathrm{DGA}_k \to \mathrm{CDGA}_k$
3. DRep_n should be a functor $\mathrm{Ho}(\mathrm{DGA_k}) \to \mathrm{Ho}(\mathrm{CDGA_k})$
4. Then $\mathrm{DRep}_n(A)$ can be defined by composition $\mathrm{Alg}_k \to \mathrm{Ho}(\mathrm{CDGA_k})$

Theorem (Berest-Khachatryan-Ramadoss, 2013)

For $A \in \mathrm{DGA}_k,\ \mathrm{Rep}_n(A) : \mathrm{CDGA}_k \to \mathrm{Sets}$ *is representable.*

(a) $\mathrm{Hom}_{\mathrm{DGA}_k}(\sqrt[n]{A}, B) \simeq \mathrm{Hom}_{\mathrm{DGA}_k}(A, M_n(B))$

(b) $\mathrm{Hom}_{\mathrm{CDGA}_k}(A_n, C) \simeq \mathrm{Hom}_{\mathrm{DGA}_k}(A, M_n(C))$

$$\sqrt[n]{-} :\ \mathrm{DGA}_k \leftrightarrows \mathrm{DGA}_k\ : M_n(-)$$
$$(-)_n :\ \mathrm{DGA}_k \leftrightarrows \mathrm{CDGA}_k\ : M_n(-)$$

Remark

The functors $\sqrt[n]{-}$ and $(-)_n$ admit right adjoint functor $M_n(-)$. We will define their left derived functors, denoted $L(\sqrt[n]{-})$ and $L(-)_n$, in the homotopy categories $\mathrm{Ho}(\mathrm{DGA}_k)$ and $\mathrm{Ho}(\mathrm{CDGA}_k)$, respectively.

Homotopy categories $\mathrm{Ho}(\mathrm{DGA}_k)$ and $\mathrm{Ho}(\mathrm{CDGA}_k)$

Theorem (Quillen, 1967)

$\mathrm{DGA}_k, \mathrm{CDGA}_k$ *are model categories.*

We recall that a *model category* is a complete and cocomplete category $\mathcal{C}$ equipped with three distinguished classes of morphisms: weak equivalences (we), fibrations (fib) and cofibrations (cof) satisfying some natural axioms.

Examples

1. Let $\mathcal{C} = \mathbf{Top}$ be the category of topological spaces.
   (a) $f \in \mathrm{we}$ iff $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for $n \geq 0$.
   (b) $f \in \mathrm{fib}$ iff f is a Serre fibration .
   (c) $f \in \mathrm{cof}$ iff f is a "nice embedding"

Examples

2. Let $\mathcal{C} = \mathbf{Com}(R)$ be the category of chain complexes over a ring R.
   (a) we := Quasi-isomorphisms (maps inducing isomorphisms on homology).
   (b) fib := Degreewise surjections ($f_n : C_n \to D_n$ is surjective for all n).
   (c) cof := Degreewise injections with projective cokernels.

One can similarly define the model category category structure on DGA_k and CDGA_k.

Definition

For a model category $\mathcal{C}$, the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ is a category with the same objects as $\mathcal{C}$, and for two objects X, Y

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(QX, QY)/\text{we}\,,$$

where QX, QY are cofibrant (quasi-free) resolutions of X, Y respectively.

Examples

1. $\mathrm{Ho}(\mathbf{Top}) \simeq$ classical homotopy category of CW complexes.
2. $\mathrm{Ho}(\mathbf{Ch}(R)) \simeq \mathbf{D}(R)$ the usual unbounded derived category.

Now are ready to formulate the main definition of BKR (2013).

Definition

$$L(-)_n : \ \mathrm{Ho}(\mathrm{DGA}_k) \to \mathrm{Ho}(\mathrm{CDGA}_k),$$
$$L(A)_n := (QA)_n,$$

where, QA is a (cofibrant) resolution of A, is called *derived representation functor* and $L(A)_n$ *derived representation scheme* .

The homology of the complex $L(A)_n$ is denoted by $H_*(DRep_n(A))$ and is called called *derived representation homology of A* .

Example (Computation of $L(A)_n$)

Let $A = \mathbb{C}[x, y]$.

1. Then $QA = \mathbb{C}\langle u, v, w\rangle$ is a free DG algebra on three generators where $\deg(u) = \deg(v) = 0, \deg(w) = 1$ and $dw = uv - vu$.

$$QA \to A,\ u \mapsto x, v \mapsto y, w \mapsto 0$$

   this induces $H_\bullet(QA) \cong A$.

2. Let u_{ij}, v_{ij} be variables of deg=0, and variables w_{ij} of deg=1. Then

$$\sqrt[n]{QA} = \mathbb{C}\langle u_{ij}, v_{ij}, w_{ij}\rangle \text{ with } \partial(w_{ij}) = (dw)_{ij} = \sum_{k=1}^{n}(u_{ik}v_{kj} - v_{ik}u_{kj})$$

$$L(A)_n = QA_n = (\sqrt[n]{QA})_{ab} = \mathbb{C}[u_{ij}, v_{ij}, w_{ij}]$$

GL_n action on $\mathrm{Rep}_n(A)$

Let $B \in \mathrm{CDGA}_k$ and $\mathrm{Rep}_n(A)(B) = \mathrm{Hom}_{\mathrm{DGA}}(A, M_n(B))$

$$\mathrm{GL}_n \times \mathrm{Rep}_n(A)(B) \to \mathrm{Rep}_n(A)(B)$$
$$(g, \rho) \mapsto \rho^g, \qquad \rho^g(a) = g\rho(a)g^{-1}$$

BKR showed that the functor of taking GL_n invariants

$$(-)_n^{\mathrm{GL}_n} : \mathrm{DGA}_k \xrightarrow{(-)_n} \mathrm{CDGA}_k \xrightarrow{(-)^{\mathrm{GL}_n}} \mathrm{CDGA}_k$$

has the left derived functor

$$L(-)_n^{\mathrm{GL}_n} : \ \mathrm{Ho}(\mathrm{DGA}_k) \to \mathrm{Ho}(\mathrm{CDGA}_k), \ L(A)_n^{\mathrm{GL}_n} := (QA)_n^{\mathrm{GL}_n}$$

BKR showed that $H_*((QA)_n^{\mathrm{GL}_n}) \cong H_*(DRep_n(A))^{\mathrm{GL}_n}$.

Example

Let $A = \mathbb{C}[x, y]$. Then recall from above

$$QA_n = \mathbb{C}[x_{ij}, y_{ij}, w_{ij}], \quad dw_{ij} = \dots .$$

The GL_n action is defined via the action on the matrices $X = (x_{ij})$, $Y = (y_{ij})$ and $W = (w_{ij})$ by conjugation. Then

$$L(A)_n^{\mathrm{GL}_n} = \mathbb{C}[x_{ij}, y_{ij}, w_{ij}]^{\mathrm{GL}_n} .$$

Remark

The description of $L(A)_n^{\mathrm{GL}_n}$ is quite complicated. For example, $L(\mathbb{C}[x, y])_4^{\mathrm{GL}_4}$ is a quotient of polynomial algebra on 32 generators and 120 relations (X. García-Martínez- E.- R.Turdibaev, Adv. Math. 2025)

Cyclic homology

Let $A \in \mathrm{DGA}_k$. The commutator quotient space

$$A_\natural = \frac{A}{[A, A]}$$

where $[A, A]$ is the super-commutator subspace. Then, we get a functor

$$(-)_\natural : \ \mathrm{DGA}_k \to \mathrm{Com}(k), \quad A \to A_\natural .$$

Theorem (Feigin-Tsygan)

The functor $(-)_\natural$ has a derived functor.

$$L(-)_\natural : \ \mathrm{Ho}(\mathrm{DGA}_k) \to \mathrm{Ho}(\mathrm{Com}_k)$$
$$A \to (QA)_\natural .$$

Moreover, for $\forall\ A \in \mathrm{Ho}(\mathrm{DGA}_k)$, $\boxed{L(A)_\natural \simeq \mathrm{CC}_\bullet(A)}$. *This implies*

$$\boxed{H_*(L(A)_\natural) \simeq HC_\bullet(A)} \ .$$

Universal representation

In (b), If we take $C = A_n$, then

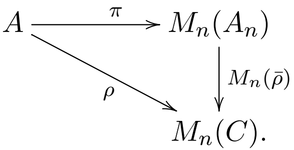
$$\mathrm{Hom}_{\mathrm{CDGA}_k}(A_n, A_n) \simeq \mathrm{Hom}_{\mathrm{DGA}_k}(A, M_n(A_n)).$$

The *universal representation* is the one corresponding to 1_{A_n} :

$$\pi_A : \; A \to M_n(A_n) \, .$$

If $\rho \in \mathrm{Hom}_{\mathrm{DGA}_k}(A, M_n(C))$, then $\exists$ unique $\bar{\rho} \in \mathrm{Hom}_{\mathrm{CDGA}_k}(A_n, C)$,

$M_n(\bar{\rho}) : \; M_n(A_n) \to M_n(C)$ so that



Derived trace

Consider $A \xrightarrow{\pi_A} M_n(A_n) \xrightarrow{\mathrm{Tr}} A_n$ which vanishes on $[A, A]$

$$\mathrm{Tr}: \ A_\natural \to A_n$$

Theorem (BKR)

For $A \in \mathrm{DGA}_k$, *the trace map* Tr *descends to a map of chain complexes*

$$\mathrm{DTr}: \ L(A)_\natural \to L(A)_n \text{ derived trace}$$

and it induces

$$\boxed{\mathrm{DTr}: \ HC_*(A) \to H_*(DRep_n(A))} \, .$$

Remark

In fact, we have $\mathrm{DTr}: \ L(A)_\natural \to L(A)_n^{\mathrm{GL}_n}$ or

$$\mathrm{DTr}: \ HC_*(A) \to H_*(DRep_n(A))^{\mathrm{GL}_n} \, .$$

Problems

(a) $\mathrm{DRep}_n(A)$ can be written explicitly, but $\mathrm{DRep}_n(A)^{\mathrm{GL}_n}$ is quite complicated.

(b) DTr is a chain map, and hence "linear". Is there any "nonlinear" generalization of this map?

A: Derived multiplicative polynomial laws.

Multiplicative polynomial laws

Let R be a commutative k-algebra. Let $\mathrm{Mod}(R)$ be the category of R-modules.

Definition

Let $M, N \in \mathrm{Mod}(R)$. A *polynomial law* $\phi : M \to N$ is a family of maps

$\{\phi_A : A \otimes_R M \to A \otimes_R N\}_{A \in \mathrm{CAlg}_k}$, such that for all $f \in \mathrm{Hom}_{\mathrm{CAlg}}(A, B)$

$$\begin{array}{ccc} A \otimes_R M & \xrightarrow{\phi_A} & A \otimes_R N \\ \downarrow{\scriptstyle f \otimes Id_M} & & \downarrow{\scriptstyle f \otimes Id_N} \\ B \otimes_R M & \xrightarrow{\phi_B} & B \otimes_R N \end{array}$$

A polynomial law $\phi : M \to N$ is called *homogeneous of degree n* if

$$\phi(au) = a^n \phi(u), \text{for all } a \in A, u \in A \otimes_R M, A \in \mathrm{CAlg}_k .$$

We denote by $\mathrm{P}^n(M, N)$ the set of all polynomial laws of degree n.

If M and N are k-algebras and

$$\phi_A(xy) = \phi_A(x)\phi_A(y)$$
$$\phi_A(1_{A\otimes M}) = 1_{A\otimes N}.$$

for all $x, y \in A \otimes_k M$, then ϕ is called *multiplicative* .

We denote by $\mathrm{MP}^n_R(M, N)$ the set of all multiplicative maps of degree n.

Examples

1. $\mathrm{MP}^1_R(A, B) = \mathrm{Hom}_{\mathrm{Alg}_k}(A, B)$.
2. $B \in \mathrm{CAlg}_k,\ \det:\ M_n(B) \to B$ is an element of $\mathrm{MP}^n_R(M_n(B), B)$.
3. If $B \in \mathrm{CAlg}_k$, then $b \mapsto b^n$ is an element of $\mathrm{MP}^n_R(B, B)$.

Divided power algebra (DPA)

Let M be an R-module. Then the divided power algebra of M, $\Gamma_R(M)$,

is a commutative algebra with identity 1_R and product $\times$, and generators

$m^{(k)}$ for all $m \in M, k \in \mathbb{Z}$, and the following relations are satisfied:

(1) $m^{(1)} = 0$, for all $i < 0$
(2) $m^{(0)} = 1_R$, for all $m \in M$
(3) $(am)^{(i)} = a^i m^{(i)}$, for all $a \in R, i \in \mathbb{N}$
(4) $(m+n)^{(k)} = \sum_{i+j=k} m^{(i)} \times n^{(j)}$, for all $k \in \mathbb{N}$
(5) $(m)^{(i)} \times (m)^{(j)} = \binom{i+j}{i} m^{(i+j)}$, for all $i, j \in \mathbb{N}$.

We write $a_1^{(i_1)} \times \cdots \times a_r^{(i_r)}$ in the form $\prod_{j=1}^r a_j^{(i_j)}$. As an R-module $\Gamma_R(M)$ is the R-linear combination of finite products $\prod_{j=1}^r a_j^{(i_j)}$.
$\Gamma_R^n(M) :=$ the submodule generated by $\{\prod_{j=1}^n a_j^{(i_j)} : \sum_{j=1}^r i_j = n\}$

Let $f: M \to N$ be R-module homomorphism. Then we define

$$\Gamma^n(f): \Gamma^n(M) \to \Gamma^n(N)$$
$$\prod_{j=1}^{n} a_j^{(i_j)} \mapsto \prod_{j=1}^{n} (a_j)^{(i_j)}.$$

So $\Gamma^n: \mathrm{Mod}(R) \to \mathrm{Alg}_R$ is a functor.

Fact. The map

$$\gamma^n: M \to \Gamma_R^n(m), \quad r \mapsto r^{(n)}$$

is a polynomial law of homogeneous of degree n.

Theorem (Roby, 1963)

$\mathrm{Hom}_{\mathrm{Mod}(R)}(\Gamma_R^n(M), N) \simeq \mathrm{P}_R^n(M, N)$ *given by* $\phi \mapsto \phi \circ \gamma^n$.

So $\Gamma_R^n(M)$ is representing object of $\mathrm{P}_R^n(M, -)$ functor.

Theorem (Roby, 1980)

If A is an R-algebra.

(i) $\Gamma^n_R(A)$ *has a structure of an R-algebra whose product is denoted by* $*$ *and is given by*

$$a^{(n)} * b^{(n)} = (ab)^{(n)} \quad \text{for all} \quad a, b \in A.$$

So, we have a functor

$$\Gamma^n_R(-):\ \mathrm{Alg}_R \to \mathrm{Alg}_R.$$

(ii) *For any* $A, B \in \mathrm{Alg}_R$, $\boxed{\mathrm{MP}^n_R(A,B) \simeq \mathrm{Hom}_{\mathrm{Alg}_R}(\Gamma^n_R(A), B)}$.

(iii) *If* $C \in \mathrm{CAlg}_R$, *then* $\boxed{\mathrm{MP}^n_R(A,C) \simeq \mathrm{Hom}_{\mathrm{Alg}_R}(\Gamma^n_R(A)^{ab}, C)}$.

Corollary

For $A \in \mathrm{Alg}_R$, $\Gamma^n_R(A)$ *and* $\Gamma^n_R(A)^{ab}$ *are the representing objects of* $\mathrm{MP}^n_R(A,-)$ *in the categories* Alg *and* CAlg_R *respectively.*

Derived divided power algebras

Let R be a commutative ring, let DGA_R and CDGA_R be the categories of DG R-algebras, and DG commutative R-algebras, repectively.

Let M be R DG-module with differential ∂ of degree $= -1$. Then the *divided power algebra* of M, denoted by $\Gamma_R(M)$ is a DG commutative R-algebra with conditions (1)-(5) above and

(6) $\partial(m^{(i)}) = (\partial m)^{(1)} \times m^{(i-1)}$, for all $i \in \mathbb{N}$.

Lemma

Let $A \in \mathrm{DGA}_k$. Then $(\Gamma_R^n(A), *, d)$ is a DG R-algebra.

Theorem

Let $A, B \in \mathrm{DGA}_R$ and $C \in \mathrm{CDGA}_R$. Then

(1) $\mathrm{DGMP}_R^n(A, B) \simeq \mathrm{Hom}_{\mathrm{DGA}_R}(\Gamma_R^n(A), B)$

(2) $\mathrm{DGMP}_R^n(A, C) \simeq \mathrm{Hom}_{\mathrm{CDGA}_R}(\Gamma_R^n(A)^{ab}, C)$

The functor

$$\begin{aligned}\Gamma_R^\bullet(-):\ \mathrm{DGMod}_R &\to \mathrm{DGMod}_R\\ M &\mapsto \Gamma_R^n(M)\end{aligned}$$

defined by Dold-Puppe in 1958.

In 1967, Quillen showed $\Gamma_R^\bullet(-)$ has the left derived functor.

Theorem (Chen-E., 2025)

The functor

$$\Gamma_k^n:\ \mathrm{DGA}_k \to \mathrm{DGA}_k, \quad A \mapsto \Gamma_k^n(A)$$

has a left derived functor

$$\begin{aligned}D\Gamma_k^n:\ \mathrm{Ho}(\mathrm{DGA}_k) &\to \mathrm{Ho}(\mathrm{DGA}_k)\\ A &\mapsto \Gamma_k^n(QA)\end{aligned}$$

where QA is a cofibrant resolution of A.

Definition

For any $A \in \mathrm{DGA}_k$, $\mathrm{D}\Gamma_k^n(A)$ is called the *derived Schur algebra*.

For $A \in \mathrm{DGA}_k$, we define

$$\mathrm{Sym}^n(A) := (A^{\otimes n})^{S_n}$$

n-th tensor symmetric algebra . Then the map

$$\Gamma^n(A) \rightarrow \mathrm{Sym}^n(A),\ x^{(n)} \mapsto \frac{x^n}{n!}$$

is an isomorphism of k-algebras when $\mathrm{char}(k) = 0$.

In invariant theory, the algebra $\mathrm{Sym}^n(A)$ for $A = End_k(V)$ is called *Schur algebra* . In that case

$$\mathrm{Sym}^n(A) \cong End_{S_n}(V^{\otimes n}) \,.$$

Derived divided power and derived representations

Let $A \in \mathrm{DGA}_R, B \in \mathrm{CDGA}_k$, and $\rho \in \mathrm{Hom}_{\mathrm{DGA}_k}(A, M_n(B))$. Then

$$\det \circ \rho \in \mathrm{MP}^n_R(A, B).$$

Since $\mathrm{MP}^n_R(A, B) \simeq \mathrm{Hom}_{\mathrm{CDGA}_R}(\Gamma^n_R(A)^{ab}, B)$, there is a unique map $\det_\rho : \ \Gamma^n_R(A)^{ab} \to B$,

$$\det \circ \rho = \det_\rho \circ \gamma^n.$$

In particular, for $\pi_A : \ A \to M_n(A_n)$

$$\underline{\det} := \det_{\pi_A} : \ \Gamma^n_R(A)^{ab} \to A_n.$$

Since $\det$ is invariant under GL_n action, we get a map

$$\underline{\underline{\det}} : \ \Gamma^n_R(A)^{ab} \to A_n^{\mathrm{GL}_n}.$$

Theorem (Chen-E, 2025)

For any $A \in \mathrm{DGA}_k$

$$\underline{\underline{\det}} : \ \mathrm{D}\Gamma^n_R(A)^{ab} \to \mathrm{DRep}_n(A)^{\mathrm{GL}_n}$$

is an isomorphism in $\mathrm{Ho}(\mathrm{DGA}_k)$.

Higher order trace map

Let $B \in \mathrm{CDGA}_k$. For a matrix $b \in M_n(B)$, let $e_i(b) := \mathrm{tr}(\wedge^i b)$, which we call the *$i$-th higher order trace* of b. It is well-known that

$$\det(\lambda \cdot I - b) = \lambda^n + \sum_{i=1}^{n} (-1)^i e_i(b) \lambda^{n-i}.$$

For any $A \in \mathrm{Ho}(\mathrm{DGA}_k)$, let us denote

$$\mathrm{CC}_\bullet^{[i]}(A) := (\mathrm{D}\Gamma_k^i(A)_\natural, \partial)$$

and call it the *i-th higher order cyclic complex* of A.
Then we obtain a well-defined chain map:

$$\mathrm{DTr}^i : \mathrm{CC}_\bullet^{[i]}(A) \to \mathrm{DRep}_n(A)^{\mathrm{GL}_n}, \quad \bar{u} \mapsto e_i(u),$$

which we call the *i-th derived higher order trace map* .

Thank You!