Classification of exchange relation planar algebras through sieving forest fusion graphs

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Joint work with Zhengwei Liu

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Subfactors and the Jones Index

Subfactor:

- A $*$-subalgebra $\mathcal{M} \subset B(H)$ is called a von Neumann algebra if it is W.O.T. closed. A factor is a vN-alg with trivial center.
- A subfactor is an inclusion of factors $\mathcal{N} \subset \mathcal{M}$.

Jones Index:

- For a subfactor $\mathcal{N} \subset \mathcal{M}$, the Jones index is defined as the dimension of $\mathcal{M}$ as a left $\mathcal{N}$-module:

$$[\mathcal{M} : \mathcal{N}] := \dim_{\mathcal{N}} \mathcal{M}.$$

- Intuitively, this measures the "relative size" of $\mathcal{M}$ compared to $\mathcal{N}$.

Jones' Index Theorem (Jones 1983):

- The index of a subfactor can only take the following values:

$$\left\{ 4\cos^2 \frac{\pi}{n} \;\middle|\; n = 3, 4, 5, \dots \right\} \cup [4, \infty].$$

- Each allowed index value can be realized by some subfactor.

Bimodule Category

Bimodule Category as a 2-Category

- objects: $\mathcal{N}$, $\mathcal{M}$;
- 1-morphisms: A–B bimodules for A, B in $\{\mathcal{N}, \mathcal{M}\}$;
- 2-morphisms: bimodule intertwiners;
- horizontal composition: relative tensor product;
- vertical composition: composition of intertwiners.

Irreducible Bimodules and Principal Graph

- Decompose $\mathcal{M}_k = \mathcal{M} \otimes_{\mathcal{N}} \mathcal{M} \otimes_{\mathcal{N}} \mathcal{M} \cdots \otimes_{\mathcal{N}} \mathcal{M}$ into irreducible bimodules.
- The principal graph is a bipartite graph between irreducible $\mathcal{N} - \mathcal{N}$ bimodules and $\mathcal{N} - \mathcal{M}$ bimodules.
- $\mathcal{N} - \mathcal{N}$ bimodule category is a tensor/fusion category.

Standard Invariants of Subfactors

- Subfactors have various invariants: index, principal graph, fusion rule, standard invariant, etc.
- Popa's deep result states that amenable subfactors are classified by the standard invariant.
- The standard invariant has various axiomatizations:
  - Ocneanu's paragroups,
  - Popa's λ-lattices,
  - Jones' subfactor planar algebras.

Classification of Subfactors

Several classification programs on subfactors have been addressed:

- Small index: ADE, Haagerup, extended Haagerup, etc
- Simple fusion rule: Bisch-Haagerup, Haagerup-Izumi (near group), integral ones, etc
- Simple generators and relations: Bisch-Jones, BMW, Liu, etc
- $\cdots$

Complexity Problem

- We need to solve certain sets of polynomial equations in all these classifications, such as flat connection equations, pentagon equations, consistency equations.
- The complexity of solving polynomial equations by the Gröbner basis grows double exponentially.
- Nowadays, we are reaching the limit of the computation power in these classification programs.
- However, new/known subfactors arisen from the classifications are quite rare.
- This is a common phenomenon for other relevant classification programs, such as fusion categories, modular tensor categories, etc.

We need brand new ideas and methods to classify/discover subfactors more efficiently.

Outline

In joint work with Zhengwei Liu [**Liu-Lu 24, arXiv:2412.17790**], we proposed a classification program based on a forest-indexed irreducible decomposition of the algebraic variety of exchange-relation and fusion-bialgebra.

It provides structure results on the classification, quantified complexity, highly efficient sieving criterions, and new examples of (subfactor) planar algebras!

Planar algebras

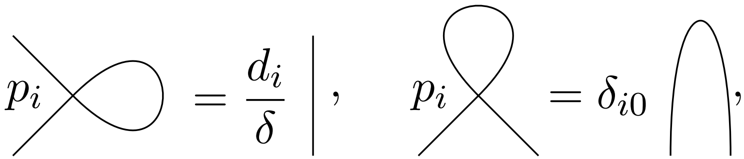
Definition (Jones 1999)

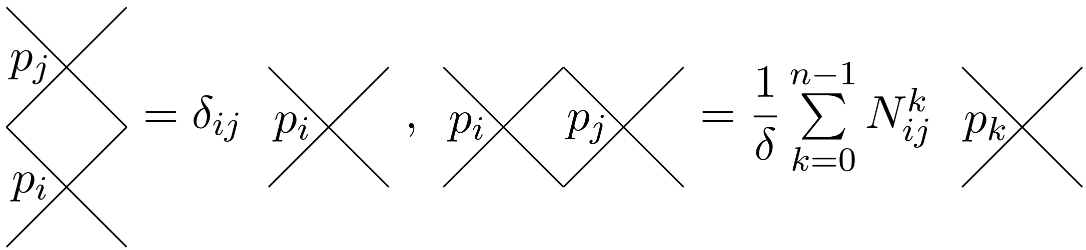
A planar algebra $\mathcal{P}$ is a family of vector spaces $(\mathcal{P}_{k,\pm})_{k\in\mathbb{N}}$, called k-box spaces, and a functor from planar tangles to those vector spaces.

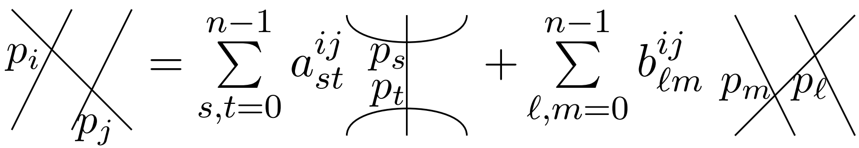
- A k-box element is presented as a 2k-valent planar diagram. Planar algebras can be defined by generators and skein relations, satisfying consistency conditions.
- From this point of view, the Jones-Temperley-Lieb planar algebras are the simplest ones with only a circle parameter δ.
- Reflection Positivity/Unitarity holds iff δ^2 is a Jones index.

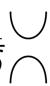
Exchange Relation

The exchange relation was introduced by Landau in 2002, inspired by the exchange relation of a biprojection given by Bisch in 1994.

1. 

2. 

3. 

Remark: $p_0 = \frac{1}{\delta}$  is the Jones projection.

Previous Classification Results

Theorem (Bisch-Jones 2000,2003)

Suppose $\mathcal{P}$ is an exchange relation planar algebra with $\dim \mathcal{P}_2 = 3$. *Then $\mathcal{P}$ is $\mathcal{P}^{\mathbb{Z}_3}$, $TL * TL$ or $\mathcal{P}^{\mathbb{Z}_2 \subset \mathbb{Z}_5 \rtimes \mathbb{Z}_2}$.*

Theorem (Liu 2016)

Suppose $\mathcal{P}$ is an exchange relation planar algebra with $\dim \mathcal{P}_2 = 4$. *Then $\mathcal{P}$ is one of the following:*

(1) $\mathcal{P}^{\mathbb{Z}_4}$ *or* $\mathcal{P}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$;

(2) $\mathcal{A} * TL$ *or* $TL * \mathcal{A}$, *where $\mathcal{A}$ is an exchange relation planar algebra with* $\dim \mathcal{A}_2 = 3$;

(3) $\mathcal{P}^{\mathbb{Z}_2} \otimes TL$;

(4) $\mathcal{P}^{\mathbb{Z}_2 \subset \mathbb{Z}_7 \rtimes \mathbb{Z}_2}$.

Jones thought that there would be many examples in the classification as the exchange relation is mild, but the outcomes were quite rare.

Three Fundamental Questions of Skein Relations

1. **Evaluation**: One needs to provide an evaluation algorithm based on enough relations, so that any closed diagram reduces to a polynomial of parameterized variables.
2. **Consistency**: The value of different reductions process should be the same, which leads to a set of polynomial equations.
3. **Unitarity**: if one wants a subfactor from the planar algebra, then one needs to verify the reflection positivity condition for certain variables.

Evaluation is ensured by exchange relations, while unitarity need to be verified on a case-by-case basis.
We first focus on the consistency condition.

Observations

Question: What are the consistency equations for exchange relations?

Theorem (Liu 16)

Exchange relation planar algebras are classified by the structure constants of operations on 2-boxes.

Question: How to express the variables $\{a_{sk}^{ij}\}$, $\{b_{sk}^{ij}\}$ in terms of $\{N_{ij}^k\}$ and $\{d_i : 0 \leq i \leq n-1\}$?
Question: How to axiomatize the structure constants $\{N_{ij}^k\}$ and $\{d_i : 0 \leq i \leq n-1\}$? (without assuming exchange relation)

Fusion bialgebra

A fusion algebra is an axiomatization of an Abelian 2-box algebra $\mathcal{P}_{2,+}$. It was introduced for unitary case in [**Liu-Palcoux-Wu, 2021**].

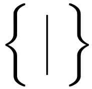
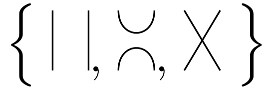
Definition (Fusion Bialgebra Variety, Liu-Lu 2024)

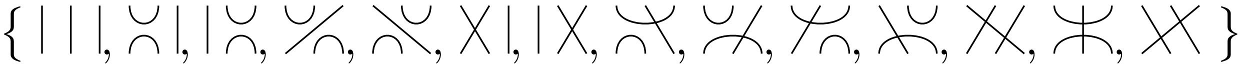
A fusion bialgebra $\mathcal{B}$ is a vector space over a field $\mathbb{K}$ with a basis $B = \{p_0, p_1, \cdots, p_{n-1}\}$, equipped with an associative multiplication $\cdot$ and convolution $*$, an involution $^-$, a linear functional $d : \mathcal{B} \to \mathbb{K}$, satisfying the following conditions:

1. $p_j \cdot p_k = \delta_{jk} p_j$;
2. $p_j * p_k = \sum_{s=0}^{n-1} N_{jk}^s p_s$;
3. $\exists$ an involution $^-$ on $\{0, 1, 2, ..., n-1\}$ s.t. $\overline{p_k} = p_{\overline{k}}$ and $\overline{p_0} = p_0$;
4. $d|_B \neq 0$ and
    - $d(p_j * p_k) = d(p_j) d(p_k)$ and $d(p_k) = d(p_{\overline{k}})$;
    - $N_{jk}^0 = \delta_{j\overline{k}} d(p_j)$;
    - $d(p_0) = 1$ and $\sum_{s=0}^{n-1} d(p_s) \neq 0$.

Exchange Relation Fusion Bialgebras

- A fusion bialgebra is called subfactorizable, if it is the 2-box space of a subfactor planar algebra.
- A exchange relation fusion bialgebra is the 2-box space of an exchange relation planar algebra.

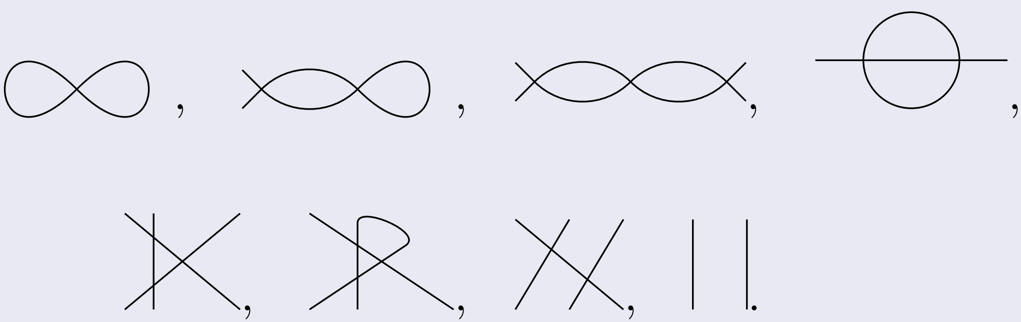
$S_0 = \{\emptyset\}$, $S_1 =$ , $S_2 =$ .

$S_3 =$ .

The 4-valent vertices are labelled by basis elements p_i in B.

Theorem (Liu-Lu 2024)

The consistency conditions for exchange relation planar algebras are equivalent to the consistency conditions on the inner product of the following tangles and elements in S_k, $k \leq 3$.



Theorem (Exchange Relation Variety, Liu-Lu 2024)

The consistency conditions for an exchange relation fusion bialgebra $\mathcal{B}$ are equivalent to the following exchange relation algebraic variety, beside the fusion bialgebra variety:

$$d_i = d_{\bar{i}}, \quad N_{kj}^{0} = \delta_{k\bar{j}} d_k, \quad N_{kj}^{i} = N_{\bar{j}\bar{k}}^{\bar{i}}, \tag{1}$$

$$d_i d_j = \sum_{s=0}^{n-1} N_{ij}^{s} d_s, \quad \sum_{s=0}^{n-1} N_{ij}^{s} N_{sk}^{\ell} = \sum_{s=0}^{n-1} N_{is}^{\ell} N_{jk}^{s}, \tag{2}$$

$$N_{kl}^{s}(a_{sk}^{ij} + b_{lk}^{ij} - \delta_{si}\delta_{\ell j}) = 0, \tag{3}$$

$$N_{zy}^{x} \sum_{s=0}^{n-1} (a_{sz}^{jk} N_{si}^{x} + b_{sz}^{jk} N_{si}^{y} - a_{s\bar{x}}^{\bar{i}j} N_{s\bar{k}}^{\bar{y}} - b_{s\bar{x}}^{\bar{i}j} N_{s\bar{k}}^{z}) = 0, \tag{4}$$

$$N_{zy}^{x} \sum_{s=0}^{n-1} (a_{sz}^{jk} N_{si}^{x} + b_{sz}^{jk} N_{si}^{y} - a_{sy}^{k\bar{i}} N_{s\bar{j}}^{\bar{z}} - b_{sy}^{k\bar{i}} N_{s\bar{j}}^{\bar{x}}) = 0, \tag{5}$$

$$N_{i\bar{j}}^{k} = \sum_{s=0}^{n-1} (a_{sk}^{ij} + b_{sk}^{ij}) d_s, \quad \delta^2 = \sum_{s=0}^{n-1} d_s. \tag{6}$$

Analogy between PA and FC

Planar Algebras	Fusion Categories
Fuison bialgebra	Fusion ring
Exchange coefficients	F-symbols
Consistency equations	Pentagon equations

Question: How to solve Consistency equations efficiently?

- The number of variables is $O(n^4)$.
- the number of equations is $O(n^6)$.
- The highest degree of the equations is 3.

Forest fusion graph

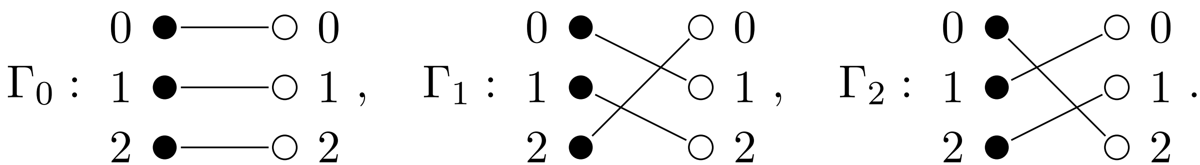
Definition

Let $\mathcal{B}$ be a fusion bialgebra with a basis $B = \{p_0, p_1, \cdots, p_{n-1}\}$. The fusion graph Γ_j of p_j is a bipartite graph with n black/white vertices labelled by elements in B. There is an edge between black p_k and white p_s if $N_{j,k}^s \neq 0$. The fusion graph $\Gamma := \{\Gamma_j : 0 \leq j \leq n-1\}$.

Theorem (Liu-Lu 2024)

A fusion bialgebra has exchange relations if and only if every fusion graph Γ_j is a forest.

Example: Forest fusion graph for the group planar algebra $\mathcal{P}^{\mathbb{Z}_3}$.



Structure Theorems

Theorem (Liu-Lu 2024)

Given a forest fusion graph Γ in an exchange relation fusion bialgebra, the exchange coefficients $\{a_{sk}^{ij}\}$, $\{b_{sk}^{ij}\}$ are 0 or ± 1 determined by Γ.

Theorem (Liu-Lu 2024)

The fusion coefficient N_{ij}^{k} is a sum of d_i determined by Γ.

$$N_{i\bar{j}}^{k} = \sum_{s=0}^{n-1} (a_{sk}^{ij} + b_{sk}^{ij}) d_s.$$

Forest Decomposition

For any forest fusion graph Γ,

- The variables $\{a_{sk}^{ij}\}$, $\{b_{sk}^{ij}\}$ and $\{N_{ij}^k\}$ reduced to quantum dimensions $\{d_i : 0 \leq i \leq n-1\}$, from $O(n^4)$ to n variables!
- Consistency equations reduce to **linear** equations and the **quadratic** associativity equation!

$$\sum_{s=0}^{n-1} N_{ij}^s N_{sk}^\ell = \sum_{s=0}^{n-1} N_{is}^\ell N_{jk}^s.$$

- The number of equations is $O(n^6)$.

Conjecture

We conjecture that this forest-indexed decomposition of the algebraic variety of exchange-relation and fusion-algebra is an irreducible decomposition.

Exponential Complexity

For any forest fusion graph Γ, the corresponding (irreducible) algebraic variety is an overdetermined system, which can be easily solved.

However, the number of fusion graphs is approximately $2^{n^3/6}$. It is about 10^{80}, the number of particles in the observable universe for $10 < n < 11$.

How to sieve forest fusion graphs that have solutions?

Table 1: The number of fusion graphs in rank n.

Rank	**# Fusion Graphs**
3	16
4	1,024
5	1,048,576
6	68,719,476,736

Free/Tensor Product Criteria

Forest fusion graph determines Free/Tensor Product structure!

Theorem (Free Product Criterion)

Let Γ be the forest fusion graph of an exchange relation fusion bialgebra $\mathcal{B}$, and let $\widetilde{\Gamma}$ be the fusion graph of the nontrivial free product of two exchange relation fusion bialgebras. If $\Gamma = \widetilde{\Gamma}$, then $\mathcal{B}$ is a nontrivial free product of two exchange relation fusion bialgebras.

Theorem (Tensor Product Criterion)

Let Γ be the forest fusion graph of an exchange relation fusion bialgebra $\mathcal{B}$, and let $\widetilde{\Gamma}$ be the fusion graph of the nontrivial tensor product of two exchange relation fusion bialgebras. If $\Gamma = \widetilde{\Gamma}$, then $\mathcal{B}$ is a nontrivial tensor product of two exchange relation fusion bialgebras.

Analytic criteria

Theorem (Associative Positivity Criterion)

Given a forest fusion graph, if there exists i, j, k, ℓ such that every monomial in $f(i,j,k,\ell)$ has the same sign, then $\mathcal{B}$ is not subfactorizable.

$$f(i,j,k,\ell) = \sum_{s=0}^{n-1} N_{ij}^{s} N_{sk}^{\ell} - \sum_{s=0}^{n-1} N_{is}^{\ell} N_{jk}^{s}.$$

Theorem (Unitary Free/Tensor Product Criterion)

A subfactorizable exchange relation fusion bialgebra $\mathcal{B}$ is a nontrivial free/tensor product if and only if the forest fusion graph is a free/tensor product.

There analytic criterion are derived from methods in quantum Fourier analysis on subfactors.

An automated classification scheme

Input: The rank n,

1. **(FF)** Enumerate all Forest Fusion graphs.
2. **(AP)** Sieve graphs that do not satisfy the Associative Positivity Criterion.
3. **(FP/TP)** Sieve graphs from the Free/Tensor Product.
4. **(GB)** Solve Consistency Equations using Gröbner Basis for each forest fusion graph.

Output: : Solutions N_{ij}^{k} and variables d_i.

Results

The running time of our algorithm on a personal computer is 1.2 seconds for $n \leq 5$ and a couple minutes for $n = 6$. The number of solutions is summarized in the following table:

n	number of graphs	algebraic solutions	subfactors
3	16	7	6
4	1,024	24	20
5	1,048,576	88	61
6	68,719,476,736	275	198+

Subfactor Case:

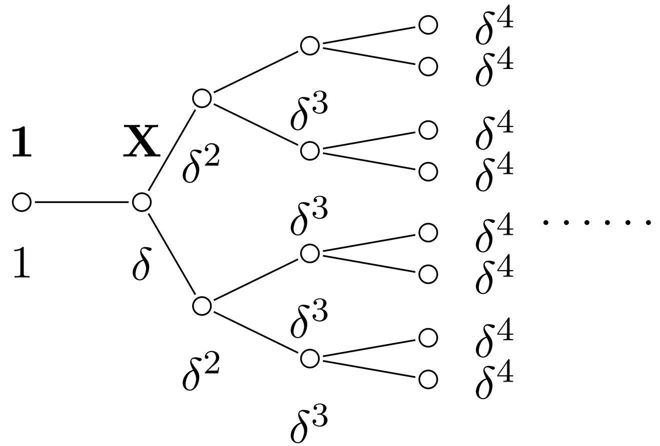
- $n = 3$, Bisch-Jones Classification (2003)
- $n = 4$, Liu Classificaton (2016)

Classification for rank 3 fusion bialgebra

Theorem (Liu-Lu 2024)

Suppose $\mathcal{B}$ is a rank 3 semisimple exchange relation fusion bialgebra over $\mathbb{C}$. Then $\mathcal{B}$ arises from one of the following planar algebras: (1) $\mathcal{P}^{\mathbb{Z}_3}$; (2) $TL * TL$; (3) $\mathcal{P}^{\mathbb{Z}_2 \subset \mathbb{Z}_5 \rtimes \mathbb{Z}_2}$; (4) The binary tree planar algebra $\mathcal{P}^B$.

Binary Tree $\mathcal{P}^B$:



Remark: $\mathcal{P}^B$ exists over any field where $\delta^2 = -1$ has a solution.

Discussion of the Results

For $n \leq 5$, all solutions are from groups, Temperley-Lieb-Jones, the binary tree planar algebras or their tensor/free product. We provide several criteria derived from Quantum Fourier Analysis to test reflection positivity. The sieving efficiency is remarkably 100%. All algebraic solutions passing the criteria produce subfactors.

For $n = 6$, there are 6 forest types which are neither groups, nor tensor/free products. They include

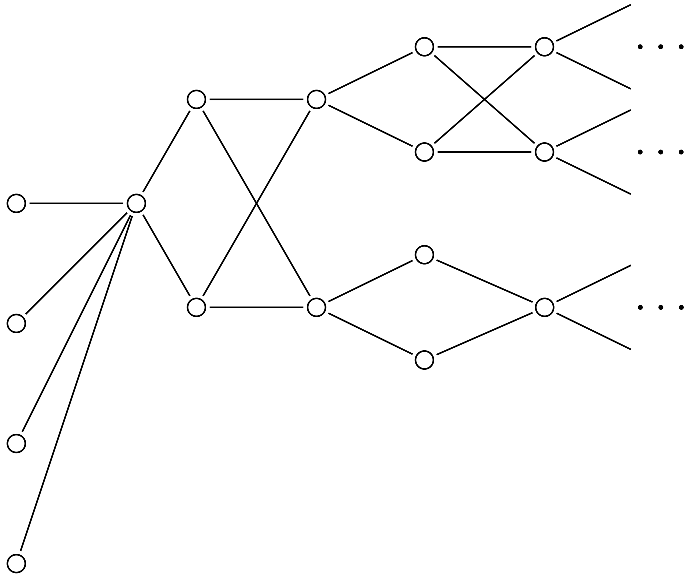
- a self dual planar algebra with index -2;
- a self dual one with index 10;
- a dual pair with index 8;
- a dual pair of one-parameter families.

New Subfactors at Rank 6

Theorem (Liu-Lu 2025)

There exists a family of infinite depth subfactors having the following graphs as principal graphs and the index values lie in the set

$$\{16\cos^2(\pi/n) : n = 5, 6, \dots\}.$$



Outlook

The rank $6 = 2 \times 3$ is the transition point that we start to see several non-trivial examples beyond (quantum) groups and tensor/free products. We expect to see more examples and new mathematical structures when n gets larger.

Using structure theorems and highly efficient sieving methods, we begin to uncover new subfactors and planar algebras in a vast and largely unexplored landscape.

We plan to design additional algorithms to further improve the efficiency of automated sieving, which we expect will lead us into a rich and wild world of subfactor planar algebras.

Thank you!

Appendix: New 6D ER fusion bialgebra ($\delta^2 = 10$)

$$N_1 \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix}, N_2 \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}, N_3 \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}$$

$$N_4 \begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}, N_5 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$d_0 = d_5 = 1, \quad d_1 = d_2 = d_3 = d_4 = 2.$$