Semisimple Yetter-Drinfel'd Hopf Algebras

Memorial
University of Newfoundland

Yorck Sommerhäuser

Theorem

Suppose that K is an algebraically closed field of characteristic zero.

Suppose that H is finite-dimensional commutative Hopf algebra over K.

Then H is a dual group ring, i.e.,

$$H \cong K[G]^*$$

for a finite group G.

Proof

H commutative $\Rightarrow S^2 = \mathrm{id}$ (H is involutory.)

Assumptions on the base field $\Rightarrow H$ is semisimple.

Wedderburn's theorem: $H \cong K \times K \times \ldots \times K = K^n$.

$\Rightarrow$ There are n distinct algebra homomorphisms to K, namely the projections to the components.

These are group-like elements in the dual.

$\Rightarrow$ These group-like elements form a basis of the dual, i.e.,

$$H^* = \mathrm{Span}(G(H^*)) \cong K[G(H^*)]$$

$\Rightarrow H \cong K[G]^*$, where $G := G(H^*)$. □

We will now investigate the same question for Yetter-Drinfel'd Hopf algebras.

Outline

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2. Yetter-Drinfel'd Hopf algebras
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Yetter-Drinfel'd modules

H: Hopf algebra

Yetter-Drinfel'd module:

Left module and left comodule over H.

Coaction: $\delta : V \to H \otimes V,\ v \mapsto v^{(1)} \otimes v^{(2)}$

Compatibility condition:

$$\delta(h.v) = h_{(1)} v^{(1)} S(h_{(3)}) \otimes h_{(2)}.v^{(2)}$$

More precisely: Left-left Yetter-Drinfel'd modules

$H = K[G]$: Yetter-Drinfel'd module = G-graded vector space with an additional G-action.

Compatibility condition:

$$\deg(v) = g \Rightarrow \deg(h.v) = hgh^{-1}$$

Quasisymmetry

Tensor product of Yetter-Drinfel'd modules:

Diagonal module and codiagonal comodule structure:

$$h.(v \otimes w) = \Delta(h).(v \otimes w) \qquad \delta(v \otimes w) = v^{(1)} w^{(1)} \otimes v^{(2)} \otimes w^{(2)}$$

$V \otimes W$ and $W \otimes V$ are isomorphic:

$$\sigma_{V,W} : V \otimes W \to W \otimes V, \;\; v \otimes w \mapsto (v^{(1)}.w) \otimes v^{(2)}$$

Note: In contrast to $v \otimes w \mapsto w \otimes v$,

$\sigma_{V,W}$ is H-linear and colinear.

Yetter-Drinfel'd Hopf algebras

Yetter-Drinfel'd Hopf algebra A over H:

Hopf algebra in the category of Yetter-Drinfel'd modules.

This means:

1. A is a (left-left) Yetter-Drinfel'd module over H.
2. A is an ordinary algebra whose product $\mu : A \otimes A \to A$ and unit map $\eta : K \to A,\ \lambda \mapsto \lambda 1$ are H-linear and colinear.
3. A is an ordinary coalgebra whose coproduct $\Delta : A \to A \otimes A$ and counit $\varepsilon : A \to K$ are H-linear and colinear.
4. A has an H-linear and colinear antipode S that satisfies the same axioms as for usual Hopf algebras.
5. and ...

The decisive difference

... Δ and ε are algebra homomorphisms.

For the counit, this does not mean anything new.

But when saying that Δ is an algebra homomorphism, we refer to the algebra structure

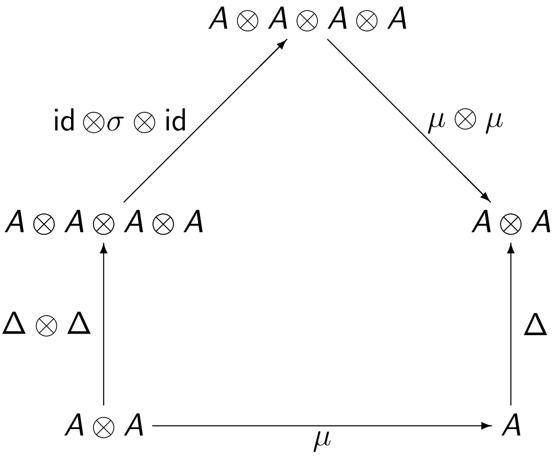
$$A \otimes A \otimes A \otimes A \xrightarrow{\mathrm{id} \otimes \sigma \otimes \mathrm{id}} A \otimes A \otimes A \otimes A \xrightarrow{\mu \otimes \mu} A \otimes A$$

on $A \otimes A$ that uses the quasisymmetry σ, and not the usual flip of the tensor factors.

This algebra structure will be denoted by $A \hat{\otimes} A$.

Diagrammatic form

This means that the following diagram commutes:



Trivial Yetter-Drinfel'd Hopf algebras

Consequence: If the quasisymmetry σ coincides with the usual flip of the tensor factors, then a Yetter-Drinfel'd Hopf algebra is an ordinary Hopf algebra.

Converse (P. Schauenburg, New York J. Math. 4 (1998)): If a Yetter-Drinfel'd Hopf algebra is an ordinary Hopf algebra, then the quasisymmetry σ coincides with the usual flip of the tensor factors.

Such Yetter-Drinfel'd Hopf algebras are called trivial.

In particular, this happens if (but not only if) the action and the coaction are both trivial.

Such Yetter-Drinfel'd Hopf algebras are called completely trivial.

Radford biproduct

If A is a Yetter-Drinfel'd Hopf algebra over H,
then $A \otimes H$ becomes a Hopf algebra in the following way:
Multiplication: Smash product

$$(a \otimes h)(a' \otimes h') = a(h_{(1)}.a') \otimes h_{(2)}h'$$

Comultiplication: Cosmash coproduct

$$\Delta_{A \otimes H}(a \otimes h) = (a_{(1)} \otimes {a_{(2)}}^{(1)} h_{(1)}) \otimes ({a_{(2)}}^{(2)} \otimes h_{(2)})$$

The Radford projection theorem (D. Radford, J. Algebra 92 (1985))

Suppose that H is a Hopf subalgebra of B,
and that $\pi : B \to H$ is a Hopf algebra retraction.

Then B decomposes as a Radford biproduct $B \cong A \otimes H$
where $A = B^{coH} = \{b \in B \mid (\mathrm{id} \otimes \pi)\Delta(b) = b \otimes 1\}$

The isomorphism is just multiplication:

$$A \otimes H \to B, \;\; a \otimes h \mapsto ah$$

Commutative semisimple Yetter-Drinfel'd Hopf algebras

From now on: K algebraically closed of characteristic zero.

We have seen: A commutative semisimple Hopf algebra is the dual group ring of a finite group.

Goal: Prove a similar structure theorem for a commutative semisimple Yetter-Drinfel'd Hopf algebra A
over the group ring $H = K[G]$ of a finite abelian group G.

So far, this has only been accomplished if $p := |G|$ is a prime.

The structure theorem in the prime order case

Suppose: $|G| = p$, an (odd) prime, c generator of G.

Theorem: If A is nontrivial, then

$$A^* \cong K^{\mathbf{Z}_p} \otimes K[H]$$

H: finite group

Algebra structure: Crossed product

Coalgebra structure: Tensor product

Note that p divides $\dim(A)$.

Data

H: Finite group

$\nu : H \to \mathbf{Z}_p^\times$: Group homomorphism

$\alpha, \beta \in Z^1(H, \mathbf{Z}_p)$: 1-cocycles

$$\alpha(st) = \alpha(s) + \nu(s)\alpha(t)$$

$q \in Z^2(H, \mathbf{Z}_p)$: 2-cocycle

ζ: Primitive p-th root of unity

Structures

e_i: Canonical basis vector in $K^p \cong K^{\mathbf{Z}_p}$

Vector space structure: $A^* = K^{\mathbf{Z}_p} \otimes K[H]$

Multiplication: Crossed product

$(e_i \otimes s)(e_j \otimes t) = \delta_{i\nu(s),j} \zeta^{iq(s,t)+\frac{i^2}{2}(\beta \cup \alpha)(s,t)} e_i \otimes st$

Comultiplication: Tensor product

Module action: $c.(e_i \otimes s) = \zeta^{i\alpha(s)} e_i \otimes s$

Comodule action: $\delta(e_i \otimes s) = c^{i\beta(s)} \otimes (e_i \otimes s)$

Antipode: $S(e_i \otimes s) = \zeta^{iq(s,s^{-1})} \zeta^{i^2\alpha(s)\beta(s)/2} e_{-i\nu(s)} \otimes s^{-1}$

Partial generalization: The triviality theorem

G: Finite abelian group

A: Yetter-Drinfel'd Hopf algebra over the group ring $K[G]$

Assumption 1: A is commutative and semisimple

Assumption 2: $\dim(A)$ and $|G|$ are relatively prime

Assertion: A is trivial

It is therefore the dual group ring $K[H]^*$ of another group H with additional structure making it a Yetter-Drinfel'd module.

Fundamental concepts in the proof

A commutative and semisimple $\Rightarrow$

A has a basis of orthogonal primitive idempotents.

Dual basis of A^*: One-dimensional characters.

Every $g \in G$ acts on A via $\phi_g : A \to A$.

This action preserves the homogeneous components.

We turn the coaction into an action of $K[G]^* \cong K[\hat{G}]$,

where $\hat{G} = \mathrm{Hom}(G, K^\times)$ is the character group $\Rightarrow$

Every $\gamma \in \hat{G}$ acts on A via $\psi_\gamma : A \to A$.

Action preserves the homogeneous components $\Rightarrow$

$$\phi_g \circ \psi_\gamma = \psi_\gamma \circ \phi_g$$

$\eta, \eta' \in A^*$ one-dimensional characters.
Define

$$T := \{g \in G \mid \phi_g^*(\eta) = \eta\} \qquad Q := \{\gamma \in \hat{G} \mid \psi_\gamma^*(\eta') = \eta'\}$$

Proposition:

$$m := |\{\phi_g^*(\eta) \mid g \in Q^\perp\}| = |\{\psi_\gamma^*(\eta') \mid \gamma \in T^\perp\}|$$

Products of characters

Usually: $\eta\eta' \in A^*$ is not again a character. Instead, we have:

Theorem: There are distinct characters $\omega_1, \ldots, \omega_m$ such that

$$\eta\eta' \in \mathrm{Span}(\omega_1, \ldots, \omega_m)$$

m is the smallest number with this property.

In addition, we have

$$\phi_g^*(\eta)\psi_\gamma^*(\eta') \in \mathrm{Span}(\omega_1, \ldots, \omega_m)$$

for all $g \in Q^\perp$ and all $\gamma \in T^\perp$. (These are m^2 characters.)

First special case: $\eta = \eta'$

Usually: $S^*(\eta)$ is not a (one-dimensional) character.

Suppose now that $\eta = \eta'$.

Define $G_\eta := Q^\perp / (T \cap Q^\perp)$.

Then we have $|G_\eta| = m$.

We call $|G_\eta|$ the index of η.

Corollary: $S^*(\eta)$ is a character $\Leftrightarrow$ The index of η is 1

Second special case

Suppose that η is a (one-dimensional) character.

Choose a (one-dimensional) character η'
that appears in the expansion of $S^*(\eta)$.

In this situation,

1. one character, say ω_1, is the counit.
2. $\text{Span}(\omega_1, \ldots, \omega_m)$ is a subalgebra of A^*.
3. It is clearly a subcoalgebra of A^*,
   because every ω_i is group-like.
4. $\text{Span}(\omega_1, \ldots, \omega_m)$ is stable under ϕ_g and ψ_γ
   for $g \in Q^\perp$ and $\gamma \in T^\perp$.
5. It is also stable under the antipode.

The core

$\mathrm{Span}(\omega_1, \ldots, \omega_m)$ is called the core of η.

It does not depend on the choice of η'

(as long as it appears in $S^*(\eta)$).

Additional property:

$$\mathrm{Span}(\omega_1\eta, \ldots, \omega_m\eta) = \mathrm{Span}(\{\phi_g^*(\eta) \mid g \in Q^\perp\})$$

$$\mathrm{Span}(\eta'\omega_1, \ldots, \eta'\omega_m) = \mathrm{Span}(\{\psi_\gamma^*(\eta') \mid \gamma \in T^\perp\})$$

So is $\mathrm{Span}(\omega_1, \ldots, \omega_m)$ a Yetter-Drinfel'd Hopf subalgebra of A^*?

No, because it is only stable under ϕ_g and ψ_γ

for $g \in Q^\perp$ and $\gamma \in T^\perp$, and not for all $g \in G$ and $\gamma \in \hat{G}$.

Theorem:

$\mathrm{Span}(\omega_1, \ldots, \omega_m)$ is a Yetter-Drinfel'd Hopf algebra over $K[G_\eta]$.

Triviality theorem: Sketch of proof

Suppose that $\dim(A)$ and $|G|$ are relatively prime.

Show first that the dimension of the core of η divides $\dim(A)$.

But the dimension of the core is the index $m = |G_\eta|$ of η, which divides $|G|$.

So the index m is equal to 1.

From here, the theorem follows with some additional work.

Back to the prime order case

For a (one-dimensional) character η,
consider the core $\mathrm{Span}(\omega_1, \ldots, \omega_m)$.

We have $m = 1$ or $m = p$.

If $m = 1$ for all η, then A is trivial as above,
so assume that $m = p$.

Then $G = G_\eta \Rightarrow G$ acts on the core.

But $\omega_1 = \varepsilon$ is a fixed point, so every ω_i must be a fixed point
$\Rightarrow$ The core is completely trivial $\Rightarrow$

$$\mathrm{Span}(\omega_1, \ldots, \omega_m) = \mathrm{Span}(\varepsilon, \omega, \omega^2, \ldots, \omega^{p-1})$$

for an invariant (and coinvariant) character ω of order p.

The quotient

Recall that the core has the additional property that

$$\mathrm{Span}(\omega_1\eta, \ldots, \omega_m\eta) = \mathrm{Span}(\{\phi_g^*(\eta) \mid g \in Q^\perp\})$$

$$\mathrm{Span}(\eta'\omega_1, \ldots, \eta'\omega_m) = \mathrm{Span}(\{\psi_\gamma^*(\eta') \mid \gamma \in T^\perp\})$$

If we pass to a quotient where $\omega = \varepsilon$, so that

$\omega_1 = \ldots = \omega_m = \varepsilon$,

the action (and also the coaction) become trivial $\Rightarrow$ The quotient is a group algebra $K[H]$.

It turns out that the entire A^* can be reconstructed from the core and the quotient:

$$A^* \cong K[\langle\omega\rangle] \otimes K[H]$$

This is the structure theorem.

An example

We have just seen:

If $|G|$ is prime, then the core of η is completely trivial.

Conjecture: The core of η is always trivial.

We now present an example where the core is trivial,
but not completely trivial.

Construction of the example

$\iota, \zeta \in K$: Fourth roots of unity, ι primitive.

A: Generated by commuting elements x and y

subject to the defining relations

$$x^4 = 1 \qquad y^2 = \frac{1}{2}(1 + \zeta x + x^2 - \zeta x^3)$$

Two automorphisms ϕ and ϕ':

$$\phi(x) := x^3 \qquad \phi(y) := x^3 y$$

$$\phi'(x) := x \qquad \phi'(y) := x^2 y$$

Basis of A

We have $\dim(A) = 8$. A basis is $\omega_1, \omega_2, \omega_3, \omega_4, \eta_1, \eta_2, \eta_3, \eta_4$:

$$\omega_1 := 1 \qquad \omega_2 := \tfrac{1}{2}(1 + \iota\zeta^2)x + \tfrac{1}{2}(1 - \iota\zeta^2)x^3$$

$$\omega_3 := \tfrac{1}{2}(1 - \iota\zeta^2)x + \tfrac{1}{2}(1 + \iota\zeta^2)x^3 \qquad \omega_4 := x^2$$

and

$$\eta_1 := y \qquad \eta_2 := x^3 y \qquad \eta_3 := x^2 y \qquad \eta_4 := xy$$

$\mathrm{Span}(\omega_1, \omega_2, \omega_3, \omega_4) \cong K[\mathbf{Z}_2 \times \mathbf{Z}_2]$:

	ω_1	ω_2	ω_3	ω_4
ω_1	ω_1	ω_2	ω_3	ω_4
ω_2	ω_2	ω_1	ω_4	ω_3
ω_3	ω_3	ω_4	ω_1	ω_2
ω_4	ω_4	ω_3	ω_2	ω_1

The coalgebra structure and the action

The coalgebra structure is determined by requiring that the basis elements are group-like:

$$\Delta(\omega_i) = \omega_i \otimes \omega_i \qquad \Delta(\eta_j) = \eta_j \otimes \eta_j$$

The group is $G := \mathbf{Z}_2 \times \mathbf{Z}_2 = \{g_1, g_2, g_3, g_4\}$, where

$$g_1 = (0,0) \qquad g_2 = (1,0) \qquad g_3 = (0,1) \qquad g_4 = (1,1)$$

$g_2 = (1,0)$ acts by ϕ and $g_3 = (0,1)$ acts by ϕ'.

$\eta_1, \eta_2, \eta_3, \eta_4$ form one G-orbit.

We have $\phi(\omega_2) = \omega_3$, so $\{\omega_2, \omega_3\}$ is a G-orbit,
while $\omega_1 = 1$ and $\omega_4 = x^2$ are fixed points.
Note that $g_3.\omega_i = \omega_i$.

The coaction on the ω_i

ω_1 and ω_4 are coinvariant:

$$\delta(\omega_1) = g_1 \otimes \omega_1 \qquad \delta(\omega_4) = g_1 \otimes \omega_4$$

Otherwise, we have

$$\begin{aligned}\delta(\omega_2) &= \tfrac{1}{2}(g_1 + g_3) \otimes \omega_2 + \tfrac{1}{2}(g_1 - g_3) \otimes \omega_3 \\ \delta(\omega_3) &= \tfrac{1}{2}(g_1 - g_3) \otimes \omega_2 + \tfrac{1}{2}(g_1 + g_3) \otimes \omega_3\end{aligned}$$

Therefore, we have

$$\begin{aligned}\sigma_{A,A}(\omega_2 \otimes \omega_i) &= \tfrac{1}{2}(g_1 + g_3).\omega_i \otimes \omega_2 + \tfrac{1}{2}(g_1 - g_3).\omega_i \otimes \omega_3 \\ &= \omega_i \otimes \omega_2\end{aligned}$$

and similarly $\sigma_{A,A}(\omega_j \otimes \omega_i) = \omega_i \otimes \omega_j$.

This means that $\mathrm{Span}(\omega_1, \omega_2, \omega_3, \omega_4)$ is trivial,

but not completely trivial.

The coaction on the η_j

On $G := \mathbf{Z}_2 \times \mathbf{Z}_2$, define a symmetric bilinear form

$$\theta : G \times G \longrightarrow K^\times$$

by requiring that

$$\begin{pmatrix} \theta((1,0),(1,0)) & \theta((1,0),(0,1)) \\ \theta((0,1),(1,0)) & \theta((0,1),(0,1)) \end{pmatrix} = \begin{pmatrix} \zeta^2 & -1 \\ -1 & 1 \end{pmatrix}$$

For $k = 1, 2, 3, 4$, we define

$$\delta(\eta_k) = \frac{1}{4} \sum_{i,j=1}^{4} \theta(g_k^{-1} g_i, g_j) g_j \otimes \eta_i$$

The antipode

$$S_A(\omega_1) = \omega_1 \quad S_A(\omega_2) = \omega_2 \quad S_A(\omega_3) = \omega_3 \quad S_A(\omega_4) = \omega_4$$

$$S_A(\eta_1) = \frac{1}{2}(\eta_1 + \frac{1}{\zeta}\eta_2 + \eta_3 - \frac{1}{\zeta}\eta_4)$$

$$S_A(\eta_2) = \frac{1}{2}(\frac{1}{\zeta}\eta_1 + \eta_2 - \frac{1}{\zeta}\eta_3 + \eta_4)$$

$$S_A(\eta_3) = \frac{1}{2}(\eta_1 - \frac{1}{\zeta}\eta_2 + \eta_3 + \frac{1}{\zeta}\eta_4)$$

$$S_A(\eta_4) = \frac{1}{2}(-\frac{1}{\zeta}\eta_1 + \eta_2 + \frac{1}{\zeta}\eta_3 + \eta_4)$$

The core of η_1

Suppose that $\eta := \eta_1$.

The formula for $S_A(\eta_1)$ shows: We can choose $\eta' = \eta_1$.

Recall from the above theorem:

There are distinct group-like elements $\omega_1, \dots, \omega_m$ such that

$$\eta\eta' \in \mathrm{Span}(\omega_1, \dots, \omega_m)$$

m is the smallest number with this property.

In addition, we have

$$\phi_g(\eta)\psi_\gamma(\eta') \in \mathrm{Span}(\omega_1, \dots, \omega_m)$$

for all $g \in Q^\perp$ and all $\gamma \in T^\perp$.

Here we have $T = \{1\}$ and $Q = \{\varepsilon\}$,
so $T^\perp = \hat{G}$ and $Q^\perp = G$.

From the defining relations:

$$\eta\eta' = \tfrac{1}{2}\omega_1 - \tfrac{\iota}{2\zeta}\omega_2 + \tfrac{\iota}{2\zeta}\omega_3 + \tfrac{1}{2}\omega_4$$

More generally:

	η_1	η_2
η_1	$\frac{1}{2}\omega_1 - \frac{\iota}{2\zeta}\omega_2 + \frac{\iota}{2\zeta}\omega_3 + \frac{1}{2}\omega_4$	$\frac{\zeta}{2}\omega_1 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_3 - \frac{\zeta}{2}\omega_4$
η_2	$\frac{\zeta}{2}\omega_1 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_3 - \frac{\zeta}{2}\omega_4$	$\frac{1}{2}\omega_1 + \frac{\iota}{2\zeta}\omega_2 - \frac{\iota}{2\zeta}\omega_3 + \frac{1}{2}\omega_4$
η_3	$\frac{1}{2}\omega_1 + \frac{\iota}{2\zeta}\omega_2 - \frac{\iota}{2\zeta}\omega_3 + \frac{1}{2}\omega_4$	$-\frac{\zeta}{2}\omega_1 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_3 + \frac{\zeta}{2}\omega_4$
η_4	$-\frac{\zeta}{2}\omega_1 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_3 + \frac{\zeta}{2}\omega_4$	$\frac{1}{2}\omega_1 - \frac{\iota}{2\zeta}\omega_2 + \frac{\iota}{2\zeta}\omega_3 + \frac{1}{2}\omega_4$

	η_3	η_4
η_1	$\frac{1}{2}\omega_1 + \frac{\iota}{2\zeta}\omega_2 - \frac{\iota}{2\zeta}\omega_3 + \frac{1}{2}\omega_4$	$-\frac{\zeta}{2}\omega_1 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_3 + \frac{\zeta}{2}\omega_4$
η_2	$-\frac{\zeta}{2}\omega_1 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_3 + \frac{\zeta}{2}\omega_4$	$\frac{1}{2}\omega_1 - \frac{\iota}{2\zeta}\omega_2 + \frac{\iota}{2\zeta}\omega_3 + \frac{1}{2}\omega_4$
η_3	$\frac{1}{2}\omega_1 - \frac{\iota}{2\zeta}\omega_2 + \frac{\iota}{2\zeta}\omega_3 + \frac{1}{2}\omega_4$	$\frac{\zeta}{2}\omega_1 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_3 - \frac{\zeta}{2}\omega_4$
η_4	$\frac{\zeta}{2}\omega_1 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_3 - \frac{\zeta}{2}\omega_4$	$\frac{1}{2}\omega_1 + \frac{\iota}{2\zeta}\omega_2 - \frac{\iota}{2\zeta}\omega_3 + \frac{1}{2}\omega_4$

So the core of $\eta = \eta_1$ is $\mathrm{Span}(\omega_1, \omega_2, \omega_3, \omega_4)$.

We have already seen that $\mathrm{Span}(\omega_1, \omega_2, \omega_3, \omega_4)$ is trivial, but not completely trivial.

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