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Automorphisms of Vertex Operator Algebras

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1. Introduction

Definition

- $V = (V, Y, \mathbf{1}, \omega)$ is a VOA. $g \in \mathrm{GL}(V)$ is called an automorphism of V if

$$g(\mathbf{1}) = \mathbf{1}, \ g(\omega) = \omega,$$

$$gY(u, z)v = Y(gu, z)gv \text{ for any } u, v \in V.$$

- $\mathrm{Aut}(V)$ is the automorphism group.
- $d \in \mathrm{gl}(V)$ is called a derivation if

$$d(\mathbf{1}) = d(\omega) = 0, \ dY(u, z)v = Y(du, z)v + Y(u, z)dv$$

- Der(V) is the derivation Lie algebra.

1. Introduction

This talk

1. The structure of Aut(V)
2. Finite automorphism group G and orbifold theory
3. If $V < A$ is a sub VOA, what is Gal(A/V)?
4. Categorical automorphisms and orbifold theory

Main result

Gal(A/V) is determined (realized) by the A-modules in the categorical setting.

2. Aut(V)

Strongly rational

- V is of CFT type if $V = \bigoplus_{n \geq 0} V_n$ with $V_0 = \mathbb{C}\mathbf{1}$.
- V is rational if V-module category is semisimple.
- V is C_2-cofinite if $\dim V/C_2(V) < \infty$ where $C_2(V) =< u_{-2}v | u, v >$.
- V is strongly rational if V is rational, C_2-cofinite and of CFT type

Theorem [Borcherds 1986]

If V is a vertex algebra then V/DV is a Lie algebra with $[u, v] = u_0 v$ for $u, v \in V_1$ where $DV =< u_{-2}\mathbf{1} | u \in V >$. In particular, If V is a vertex operator algebra of CFT type then V_1 is a Lie algebra.

2. Aut(V)

Theorem [Dong-Mason 2004]

If V is strongly rational then V_1 is a reductive Lie algebra.

Conjecture 1

If V is rational and of CFT type then $\mathrm{Der}(V) = V_1$.

Remark

This conjecture is similar to a result in Lie theory: If $\mathfrak{g}$ is a finite dimensinal semisimple Lie agebra then $\mathrm{Der}(\mathfrak{g}) = \mathfrak{g}$.

2. Aut(V)

Theorem [Dong-Griess 2002]

If V is finitely generated then Aut(V) is an algebraic group.

Theorem [Dong-Zhang 2008]

If V is rational then V is finitely generated. In particular, Aut(V) is an algebraic group.

We now assume that V is strongly rational. Let N be the subgroup of Aut(V) generated by e^{u_0} for $u \in V_1$. Then N is a normal subgroup of Aut(V).

Conjecture 2

If V is rational and of CFT type then $\mathrm{Aut}(V)_e = N$ where $\mathrm{Aut}(V)_e$ is the maximal connected component of Aut(V) containing the identity.

2. Aut(V)

Remark

- Conjecture 2 implies that $\mathrm{Aut}(V)/N$ is a finite group.
- Conjecture 1 and Conjecture 2 are equivalent.
- Conjecrure 2 implies that $|\mathrm{Aut}(V)| < \infty$ if $V_1 = 0$.
- Conjecture 2 on $\mathrm{Aut}(V)$ is open in general.
- Conjectures 1 and 2 are false if V is not rational.
- Counter example: Let V be the Heisenberg VOA. Then $\mathbb{C}[\alpha(-n)|\alpha \in H, n > 0]$ where H is a finite dimensional vector space with a nondegenerate symmetric bilinear form. V is irrational and $\mathrm{Aut}(V) = O(H)$ is an infinite group and $N = \{1\}$.

2. Aut(V)

Examples

- Moonshine VOA $V^\natural$ constructed by Frenkel-Lepowsky-Meurman (1988) is strongly rational (Dong 1994) and $V_1^\natural = 0$. Aut($V^\natural$) is the Monster simple group (Borcherds 1986, FLM 1988).
- Affine VOA $L_{\mathfrak{g}}(k,0)$ associated to finite dimensional simple Lie algebra is strongly rational (Frenkel-Zhu 1992). $\mathrm{Aut}(L_{\mathfrak{g}}(k,0)) \cong \mathrm{Aut}(\mathfrak{g})$ and $|\mathrm{Aut}(L_{\mathfrak{g}}(k,0))/N| \leq 3$.
- Conjecrure 2 is also true for lattice VOA V_L associated with any positive definite even lattice L (Dong-Nagatomo 1999).

3. Orbifold theory

Orbifold theory

- V is a vertex operator algebra.
- G is a finite automorphism group of V.
- Orbifold theory: Study the V^G-module category.

Twisted modules

- Main feature: Appearance of g-twisted V-module: A g-twisted V-module is not a V-module but restricts to a V^G-module.
- Twisted modules are non local modules in category theory (will be discussed further later).

3. Orbifold theory

Orbifold theory conjecture

(1) If V is rational then V^G is rational.
(2) Any irreducible V^G-module appears in an irreducible g-twisted V-module for some $g \in G$ [Dijkgrakk-Vafa-Verlinde-Verlinde 1990].

Stronger conjecture

If U is a conformal subalgebra of V such that V is a finite sum of irreducible U-modules. Then U is rational iff V is rational

- (Carnahan-Miyamoto 2016) Conjecture (1) is true if G is solvable and V is C_2-cofinite
- (McRae 2019) Conjecture (1) is true if V^G is C_2-cofinite

3. Orbifold theory

Theorem [Dong-Ren-Xu 2017]

If V^G strongly rational then every irreducible V^G-module appears in an irreducible g-twisted V-module for some $g \in G$. That is, Conjecture (2) follows from Conjecture (1). In particular, if G is solvable, Orbifold theory Conjecture is solved completely.

3. Orbifold theory

V is called holomorphic if V is strongly rational and V is the only irreducible V-module.

Conjecture[Dijkgraaf-Witten, Dijkgraaf-Pasquier-Roche 1990]

If V is holomorphic VOA and G is a finite automorphism group of V then there exists $\alpha \in H^3(G, U(1))$ such that $\mathcal{M}_{V^G}$ is braidedly equivalent to the module category of twisted Drinfeld double $D^\alpha[G]$.

Theorem [Dong-Ng-Ren 2025]

Dijkgraaf-Witten, Dijkgraaf-Pasquier-Roche conjecture is true. Moreover, $\{\mathcal{M}_{V^G} | V \in H\}$ form a group isomorphic to $H^3(G, U(1))$ with product $\mathcal{M}_{V^G} \cdot \mathcal{M}_{U^G} = \mathcal{M}_{(V \otimes U)^G}$ where H is the collection of all holomorphic VOAs V such that $G < \mathrm{Aut}(V)$.

4. Gal(A/V)

Setting

- A is a VOA.
- V is a sub VOA such that A is a finite sum of irreducible V-modules.

Question: What can we say about Gal(A/V)?

Theorem [Schur 1911]

If G is a subgroup of $GL_n(\mathbb{C})$ such that every element of G has a finite order then G is locally finite (any finitely generated subgroup is a finite group).

4. Gal(A/V)

Theorem [Dong-Ng-Ren-Xu 2025]

1. Gal(A/V) is locally finite.
2. If V is strongly rational, then Gal(A/V) is a finite group. In fact, we know how to determine each $g \in$ Gal(A/V) explicitly.

Remark

- The proof of (1) uses Schur's Theorem.
- The proof of (2) uses the fusion action (see Li Ren's talk).
- These results hold in (follow from) categorical orbifold setting.

5. Categorical automorphisms

Notations

- $\mathcal{F}$: Fusion category
- $\mathrm{Irr}(\mathcal{F})$: equivalence classes of simple objects
- $K(\mathcal{F})$: the fusion algebra over $\mathbb{C}$, which is a semisimple associative algebra
- $\mathcal{C}$: modular tensor category (MTC)

Condensable algebra

$A \in \mathcal{C}$ is called a condensable algebra:

- A is an algebra: $m_A : A \otimes A \to A$, $u_A : \mathbf{1} \to A$
- A is connected: $\dim \mathcal{C}(\mathbf{1}, A) = 1$
- A is commutative: $m_A = m_A R_{A,A}$ where $R_{A,A} : A \otimes A \to A \otimes A$ is the braidng

5. Categorical automorphisms

Examples

- If G is a finite group, then $\mathbb{C}[G]^*$ is a condensable algebra in Rep(G) (which is a symmetric braided fusion category but not a MTC). One can define condensable algebra in any braided fusion category.
- If A is a VOA and V is strong rational conformal sub-VOA. Then A is a condensable algebra in $\mathcal{M}_V$.

5. Categorical automorphisms

Let A be a condensable algebra in an MTC $\mathcal{C}$.

Automorphisms

- An isomorphism $\sigma \in \mathcal{C}(A, A)$ is called an automorphism of A if $\sigma u_A = u_A$ and $m_A(\sigma \otimes \sigma) = \sigma m_A$.
- The set of all automorphisms of A is denoted by $\mathrm{Aut}_{\mathcal{C}}(A)$.
- If V is strong rational, A is an extension of V, $\mathcal{C} = \mathcal{M}_V$, then $\mathrm{Aut}_{\mathcal{C}}(A) = \mathrm{Gal}(A/V)$.

Theorem [Dong-Ng-Ren-Xu 2025]

- $\mathrm{Aut}_{\mathcal{C}}(A)$ is a finite group.
- If V is strong rational, A is an extension of V, then $\mathrm{Gal}(A/V)$ is a finite group.

5. Categorical automorphisms

Let A be a condensable algebra in a MTC $\mathcal{C}$ and we now determine $\mathrm{Aut}_{\mathcal{C}}(A)$ explicitly.

A-modules

- $M \in \mathcal{C}$ is a left A-module: $m_M : A \otimes M \to M$.
- A-module category $\mathcal{C}_A$ is a fusion category.
- Each $X \in \mathcal{C}_A$ has a dimension $\dim_A(X)$. For each subcategory $\mathcal{D}$ one can define

$$\dim(\mathcal{D}) = \sum_{X \in \mathrm{Irr}(\mathcal{D})} d_A(X)^2.$$

- Local A-module category $\mathcal{C}_A^0$ is a subcategory of $\mathcal{C}_A$ and is a MTC.

5. Categorical automorphisms

A-modules

- $\mathcal{C}_A$ is a $\mathcal{C}_A^0$-module.
- $$\mathcal{C}_A = \bigoplus_{i=1}^{k} (\mathcal{C}_A)_i, \mathrm{Irr}(\mathcal{C}_A) = \bigcup_{i=1}^{k} \mathrm{Irr}((\mathcal{C}_A)_i)$$
  is the indecomposable $\mathcal{C}_A^0$-modules decomposition of $\mathcal{C}_A$ with $(\mathcal{C}_A)_1 = \mathcal{C}_A^0$.
- For each i set
  $$g_i := \frac{1}{\dim(\mathcal{C}_A^0)} \sum_{X \in \mathrm{Irr}((\mathcal{C}_A)_i)} d_A(X) X$$

5. Categorical automorphisms

Theorem [Dong-Ng-Ren-Xu 2025]

- There is an action of $K(\mathcal{C}_A)$ (fusion action) on A such that Schur-Weyl duality holds:

$$A = \oplus_{x \in \mathrm{Irr}(\mathcal{C})} W_x \otimes x$$

where the mutiplicity space $W_x = \mathrm{Hom}_{\mathcal{C}}(x, A)$ is an irreducible $K(\mathcal{C}_A)$-module, and $W_x \cong W_y$ iff $x = y$.

- The kernel of this action is $(1 - g_1)K(\mathcal{C}_A)$ and

$$K(\mathcal{C}_A) = g_1 K(\mathcal{C}_A) + (1 - g_1)K(\mathcal{C}_A).$$

- The set $B = \{g_i \mid i = 1, \ldots, k\}$ is a basis of $g_1 K(\mathcal{C}_A)$.
- If $\mathcal{C}$ is pseudounitry, $\mathrm{Aut}_{\mathcal{C}}(A)$ is a finite group. In fact,

$$\mathrm{Aut}_{\mathcal{C}}(A) = \{g_i | \dim((g_i) = 1\}.$$

6. Categorical orbiford theory

Let A be a condensable algebra in an MTC $\mathcal{C}$.

Twisted A-module

- Let $g \in \mathrm{Aut}_{\mathcal{C}}(A)$. $X \in \mathcal{C}_A$ is called a g-twisted A-module if

$$m_A = m_A(g \otimes 1) R_{X,A} R_{A,X} : A \otimes X \to X$$

where

$$m_X : A \otimes X \to X$$

defines the A-module structure of X in $\mathcal{C}_A$ and

$$R_{A,X} : A \otimes X \to X \otimes A$$

is the braiding isomorphism.

- In the case where A is a VOA, these two definitions of g-twisted modules coincide.

6. Categorical orbiford theory

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Let G be a finite group.

G-extension

- $\mathcal{C}_A$ is called a G-extension of $\mathcal{C}_A^0$ if

$$\mathcal{C}_A == \bigoplus_{g\in G} \mathcal{C}_A(g), \ \mathrm{Irr}(\mathcal{C}_A) = \bigcup_{g\in G} \mathrm{Irr}(\mathcal{C}_A(g))$$

  such that $\mathcal{C}_A(1) = \mathcal{C}_A^0$ and $\mathcal{C}_A(g) \otimes \mathcal{C}_A(h)$ is a subcategory of $\mathcal{C}_A(gh)$.

6. Categorical orbiford theory

Theorem [Dong-Ng-Ren-Xu 2025]

- If $\mathcal{C}_A$ is a G-extension of $\mathcal{C}_A^0$, then $\mathrm{Aut}_{\mathcal{C}}(A) = G$ and $\mathcal{C}_A(g)$ is the g-twisted A-module category of $\mathcal{C}_A$ and $A^G = \mathbf{1}$.
- If $G = \mathrm{Aut}_{\mathcal{C}}(A)$ such that $A^G = \mathbf{1}$. Then $\mathcal{C}_A = \bigoplus_{g\in G} \mathcal{C}_A(g)$ is a G-extension of $\mathcal{C}_A^0$ where $\mathcal{C}_A(g)$ is the g-twisted A-module category.
- Statements $G = \mathrm{Aut}_{\mathcal{C}}(A)$ and $\mathcal{C}_A$ is a G-extension of $\mathcal{C}_A^0$ are equivalent.

Remark

Kirillov (2002) and McRae (2021) studied twisted modules in category setting. In fact, McRae proved the tesnor product of a g-twisted module and an h-twisted module is a gh-twisted module.

THANKS