New classes of Quasi-Hopf algebras

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International Conference Hopf Algebras and Tensor Categories
TSIMF Sanya China, January 19-23, 2026

(joint work with D. Bulacu and D. Popescu)

Contents

Preliminaries

- We characterise double biproducts as ordinary biproducts, and show that their deformations by 2-cocycles are double wreath quasi-quantum groups.
- We present examples of 2-cocycles from almost skew pairings in categories of Yetter-Drinfeld modules and show that various types of quasi-quantum groups known in the literature are of this type.
- We define a quasi-Hopf analogue of the Drinfeld-Jimbo quantum groups $U_q(\mathfrak{g})$.

Bespalov and Drabant showed that the Majid's double biproduct construction has a deep categorical nature.

1 Bespalov & Drabant, *J. Pure Appl. Alg.* **123** (1998), 105–129.

2 Majid, *Math. Proc. Camb. Phil. Soc.* **125** (1999), 151–192.

H a quasi-Hopf algebra

P. Schauenburg (2012) shows that there is a strongly monoidal equivalence between ${}_H^H\mathcal{M}_H^H$ and ${}_H^H\mathcal{YD}$, the Drinfeld center of ${}_H\mathcal{M}$

The following three notions are equivalent:

1. a braided Hopf algebra in ${}_H^H\mathcal{M}_H^H$
2. a left biproduct quasi-Hopf algebra $C \times H$ for some Hopf algebra C in ${}_H^H\mathcal{YD}$
3. a quasi-Hopf algebra A with a projection $\pi : A \to H$.

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- The algebras in ${}_H^H\mathcal{M}_H^H$ are (left) smash product algebras $C\#H$ between an algebra C in ${}_H^H\mathcal{YD}$ and H.
- $C\#H$ is the k-vector space $C \otimes H$ endowed with the multiplication given by

$$(c\#h)(c'\#h') = (x^1 \cdot c)(x^2 h_1 \cdot c')\#x^3 h_2 h',$$

for all $c, c' \in C$ and $h, h' \in H$ and unit $1_{C\#H} = 1_C\#1_H$. We need only C an algebra in ${}_H\mathcal{M}$, the monoidal category of left H-modules, in order to get $C\#H$ a k-algebra; the algebra structure of C in ${}_H^H\mathcal{YD}$ is needed to regard $C\#H$ as an algebra in ${}_H^H\mathcal{M}_H^H$.

Some structures

- The coalgebras in ${}^H_H\mathcal{M}^H_H$ are (left) smash product coalgebras $C \rtimes H$ between a coalgebra C in ${}^H_H\mathcal{YD}$ and H, a coalgebra within the monoidal category ${}_H\mathcal{M}_H$ of H-bimodules.
- $C \rtimes H = C \otimes H$ as k-vector spaces, with comultiplication determined by

$$\Delta(c \rtimes h) = (y^1X^1 \cdot c_{\underline{1}} \rtimes y^2Y^1(x^1X^2 \cdot c_{\underline{2}})_{\{-1\}}x^2X^3_1h_1) \otimes(y^3_1Y^2 \cdot (x^1X^2 \cdot c_{\underline{2}})_{\{0\}} \rtimes y^3_2Y^3x^3X^3_2h_2),$$

for all $c \in C$ and $h \in H$, and counit $\varepsilon_{C\rtimes H} = \underline{\varepsilon}_C \otimes \varepsilon_H$.

Explicit structures

- Thus, any bialgebra (resp. Hopf algebra) in ${}_H^H\mathcal{M}_H^H$ is of the form $C \otimes H$ for a certain bialgebra (resp. Hopf algebra) C in ${}_H^H\mathcal{YD}$, and is denoted by $C \times H$.
- $C \times H = C\#H$ as an algebra, $C \times H = C \rtimes H$ as a coalgebra and, moreover, it is a quasi-bialgebra (resp. quasi-Hopf algebra) with reassociator (resp. antipode) defined by

$$\Phi_{C\times H} = 1_C \times X^1 \otimes 1_C \times X^2 \otimes 1_C \times X^3,$$
$$(s^l(c \times h) = (1_C \times S(X^1 p_1^1 c_{\{-1\}} h)\alpha)(X^2 p_2^1 \cdot S_C(c_{\{0\}}) \times X^3 p^2),$$
$$1_C \times \alpha, 1_C \times \beta),$$

for all $c \in C$, $h \in H$, where we wrote $c \times h$ in place of $c \otimes h$ in order to distinguish the quasi-bialgebra structure on $C \otimes H$ given by the left biproduct construction.

The right handed version

- The algebras in ${}_H^H\mathcal{M}_H^H$ are the right smash product algebras $H\#B$ between an algebra B in $\mathcal{YD}_H^H$ and H, where $H\#B$ is $H \otimes B$ equipped with multiplication and unit given by, $\forall\ b,\ b' \in B,\ h,\ h' \in H,$

$$(h\#b)(h'\#b') = hh'_1x^1\#(b \cdot h'_2x^2)(b' \cdot x^3),\ \ 1_{H\times B} = 1_H \times 1_B$$

- The coalgebras in ${}_H^H\mathcal{M}_H^H$ are the right smash product coalgebras $H \rtimes B$ between a coalgebra B in $\mathcal{YD}_H^H$ and H, where $H \ltimes B$ is $H \otimes B$ endowed with the comultiplication and counit determined by

$$\Delta(h \ltimes b) = (h_1X_1^1x^1Y^1y_1^1 \ltimes (b_{\underline{1}} \cdot X^2x^3)_{(0)} \cdot Y^2y_1^3) \\ \otimes(h_2X_2^1x^2(b_{\underline{1}} \cdot X^2x^3)_{(1)}Y^3y^2 \ltimes b_{\underline{2}} \cdot X^3y^3),\ \ \varepsilon_{H\times B} = \varepsilon_H \otimes \underline{\varepsilon}_B;$$

- $H \times B$ is the right biproduct of B, H, a quasi-Hopf algebra with

$$\Phi_{H\times B} = X^1 \times 1_B \otimes X^2 \times 1_B \otimes X^3 \times 1_B,$$
$$(s^r(h \times b) = (\tilde{q}^1 X^1 \times S_B(b_{(0)}) \cdot \tilde{q}^2_1 X^2)(\beta S(hb_{(1)}\tilde{q}^2_2 X^3) \times 1_B),$$
$$\alpha \times 1_B, \beta \times 1_B).$$

- Let $B \in \mathcal{YD}_H^H$. Define $\overline{B} \in {}_H^H\mathcal{YD}$ as the object B with structure given by

$$h \cdot b = b \cdot S^{-1}(h) \ \text{ and}$$
$$\lambda_{\overline{B}}(b) = g^1 S((b \cdot S^{-1}(f^1))_{(1)}) f^2 \otimes (b \cdot S^{-1}(f^1))_{(0)} \cdot S^{-1}(g^2).$$

- If B has an algebra structure in $\mathcal{YD}_H^H$ then $\overline{B}$ is an algebra in ${}_H^H\mathcal{YD}$ with multiplication

$$b \bullet b' = (b \cdot S^{-1}(f^1))_{(0)}[b' \cdot S^{-1}(f^2)(b \cdot S^{-1}(f^1))_{(1)}],$$

  for all $b, b' \in B$, and unit equals 1_B, the unit of B (the juxtaposition denotes the multiplication of B in $\mathcal{YD}_H^H$).
- If B is a coalgebra in $\mathcal{YD}_B^B$ then $\overline{B}$ is a coalgebra in ${}_H^H\mathcal{YD}$ with counit equals $\underline{\varepsilon}_B$ and comultiplication defined, for all $b \in B$, by

$$\underline{\Delta}_{\overline{B}}(b) = b_{\overline{1}} \otimes b_{\overline{2}} := (b_{\underline{1}})_{(\underline{0})} \cdot X^2 p_2^1 S^{-1}(g^1) \otimes b_{\underline{2}} \cdot S^{-1}(g^2 S(X^1 p_1^1)(b_{\underline{1}})_{(\underline{1})} X^3 p^2)$$

  where $(\underline{\Delta}_B : B \ni b \mapsto b_{\underline{1}} \otimes b_{\underline{2}} \in B \otimes B, \underline{\varepsilon}_B)$ is the coalgebra structure of B in $\mathcal{YD}_H^H$.

The isomorphism

- If B is a (co)algebra in $\mathcal{YD}_H^H$ then $\overline{B}$ is a (co)algebra in ${}_H^H\mathcal{YD}$ and the smash product algebras $H\#B$ and $\overline{B}\#H$ are isomorphic.
- A right biproduct quasi-bialgebra (resp. quasi-Hopf algebra) is always isomorphic to a left biproduct quasi-bialgebra (resp. quasi-Hopf algebra).
- In any of these situations the isomorphism is given by $\overline{\nu}_B$ defined by

$$\overline{\nu}_B(b\otimes h) = q^1 g^1 S(q_2^2 g_2^2 b_{(\underline{1})_2}\tilde{p}^2) f^1 h_1 \otimes b_{(\underline{0})} \cdot S(q_1^2 g_1^2 b_{(\underline{1})_1}\tilde{p}^1) f^2 h_2.$$

- Its inverse is

$$\overline{\nu}_B^{-1}(h\otimes b) = (b\cdot x^3)_{(0)} \cdot \tilde{p}^1 S^{-1}(h_1 x^1) \otimes h_2 x^2 (b\cdot x^3)_{(1)} \tilde{p}^2.$$

The natural condition

- Assume further that B, C satisfy the compatibility relation

$$b \otimes c = b_{(0)} \cdot c_{\{-1\}} \otimes b_{(1)} \cdot c_{\{0\}}, \ \ \forall\, b \in B,\ c \in C.$$

- It is imposed by the fact that: since $Y = C \otimes H$ and $X = H \otimes B$ are bialgebras (resp. Hopf algebras) in ${}^H_H\mathcal{M}^H_H$, the tensor product algebra and coalgebra structure on $\underline{Z} := Y \otimes_H X$ afford a bialgebra (resp. Hopf algebra) structure on $\underline{Z}$ in ${}^H_H\mathcal{M}^H_H$ if and only if $d_{Y,X} \circ d_{X,Y} = \mathrm{Id}_{X \otimes_H Y}$, where d is the braiding of ${}^H_H\mathcal{M}^H_H$.
- Let X, Y be the objects of ${}^H_H\mathcal{M}^H_H$ defined by the bialgebras $C \in {}^H_H\mathcal{YD}$, and respectively $B \in \mathcal{YD}^H_H$, as in the above. Then the following assertions are equivalent:
  (i) $\underline{Z} = Y \otimes_H X$ is a bialgebra in ${}^H_H\mathcal{M}^H_H$;
  (ii) $C \widetilde{\otimes} \overline{B}$ is a bialgebra in ${}^H_H\mathcal{YD}$;
  (iii) For all $c \in C$ and $b \in B$ the preceding relation holds.

Double biproduct quasi-Hopf algebras

- $\underline{Z} = Y \otimes_H X \equiv Z := C \otimes H \otimes B$ is a k-algebra with multiplication

$$(c\otimes h\otimes b)(c'\otimes h'\otimes b') = (y^1\cdot c)(y^2h_1x^1\cdot c')\otimes y^3h_2x^2h'_1z^1\otimes(b\cdot x^3h'_2z^2)(b'\cdot z^3).$$

  The unit of Z is $1_C \otimes 1_H \otimes 1_B$, i.e. as a k-algebra $Z = C\#H\#B$, the two-sided smash product algebra of C, B and H[1].

- Z is an H-bimodule coalgebra with comultiplication given by

$$\begin{aligned}\Delta_Z(c\otimes h\otimes b) &= [(c \bowtie h)_1\cdot Y_1^1t^1Z^1u_1^1\otimes(b_{\underline{1}}\cdot Y^2t^3)_{(0)}\cdot Z^2u_2^1]\\ &\quad\otimes[(c\bowtie h)_2\cdot Y_2^1t^2(b_{\underline{1}}\cdot Y^2t^3)_{(1)}Z^3u^2\otimes b_{\underline{2}}\cdot Y^3u^3]\\ &= [y^1X^1\cdot c_{\underline{1}}\otimes y^2T^1(z^1X^2\cdot c_{\underline{2}})_{\{-1\}}z^2X_1^3\cdot(h\bowtie b)_1]\\ &\quad\otimes[y_1^3T^2\cdot(z^1X^2\cdot c_{\underline{2}})_{\{0\}}\otimes y_2^3T^3z^3X_2^3\cdot(h\bowtie b)_2],\end{aligned}$$

  and counit $\varepsilon_Z = \underline{\varepsilon}_C\otimes\varepsilon_H\otimes\underline{\varepsilon}_B$.

[1] Bulacu, Panaite, Van Oystaeyen, Comm. Math. Phys. 266 (2006)

Double biproduct quasi-Hopf algebras

- We denote this coalgebra structure on Z by $C \rtimes H \ltimes B$.
- $C \times H \times B := C\#H\#B$ as an algebra.
- $C \times H \times B := C \rtimes H \rtimes B$ as a coalgebra.
- $C \times H \times B$ is a quasi-bialgebra with reassociator $1_C \times \Phi \times 1_B$, and
- a quasi-Hopf algebra with antipode

$$\begin{aligned} s(b \times h \times c) = {} & (1_C \times S(Y^1(z^1x^1 \cdot c)_{\{-1\}}z^2x_1^2h_{(1,1)}y_1^1X_1^1t^1p^1)\alpha \times 1_B) \\ & (\underline{S}_C(Y^2 \cdot (z^1x^1 \cdot c)_{\{0\}}) \times Y^3z^3x_2^2h_{(1,2)}y_2^1X_2^1t^2 \\ & \times \underline{S}_B((b \cdot y^3)_{(0)} \cdot X^3t^3))(1_C \times p^2S(x^3h_2y^2(b \cdot y^3)_{(1)}X^3) \times 1_B) \end{aligned}$$

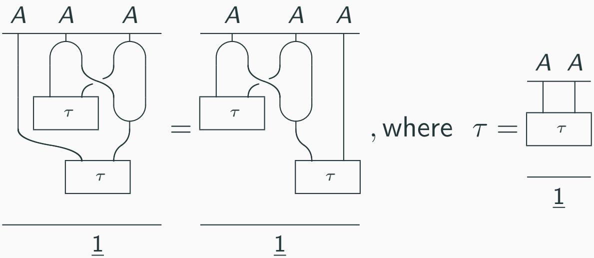
and distinguished elements $1_C \times \alpha \times 1_B$ and $1_C \times \beta \times 1_B$.

Double biproducts are biproducts

- $C \times H \times B \equiv (C\widetilde{\otimes}\overline{B})\#H$ as an algebra;
- $C \times H \times B \equiv (C\widetilde{\otimes}\overline{B}) \bowtie H$ as a coalgebra;
- $C \times H \times B \equiv (C\widetilde{\otimes}\overline{B}) \times H$ as a quasi-Hopf algebra.
- In all these cases the isomorphism is produced by χ defined by

$$\chi(c\otimes h\otimes b) = (y^1\cdot c\widetilde{\otimes}(b\cdot x^3)_{(0)}\cdot \tilde{p}^1 S^{-1}(y^2 h_1 x^1))\otimes y^3 h_2 x^2 (b\cdot x^3)_{(1)}\tilde{p}^2.$$

- A 2-cocycle of a braided bialgebra A in $(\mathcal{C}, c)$ is a morphism $\tau : A \otimes A \to \underline{1}$ in $\mathcal{C}$ obeying $\tau(\underline{\eta}_A \otimes \mathrm{Id}_A) = \underline{\varepsilon}_A = \tau(\mathrm{Id}_A \otimes \underline{\eta}_A)$ and



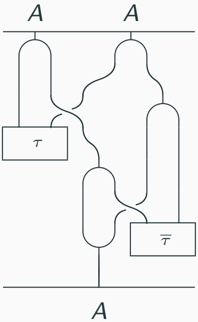
and for simplicity we assumed $\mathcal{C}$ strict monoidal.

- A 2-cocycle τ of a bialgebra A in $(\mathcal{C}, c)$ is called invertible if it is convolution invertible

2-**cocycles in braided categories**

- Let τ be an invertible 2-cocycle of the bialgebra A.
- A_τ is the coalgebra A with unit $\underline{\eta}_A$ and multiplication

$\underline{m}_A^\tau :=$



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- A_τ is a bialgebra in $\mathcal{C}$ and, moreover, a Hopf algebra with $\underline{S}_A^\tau := (u_\tau \otimes \underline{S}_A \otimes u_{\overline{\tau}})(\mathrm{Id}_A \otimes \underline{\Delta}_A)\underline{\Delta}_A$, provided that so is A $\underline{S}_A$;
- $u_\tau := \tau(\mathrm{Id}_A \otimes \underline{S}_A)\underline{\Delta}_A : A \to \underline{1}$, and similar for $u_{\overline{\tau}}$.

Theorem
$(\mathcal{F}, \varphi_2, \varphi_0) : (\mathcal{C}, c) \to (\mathcal{D}, d)$ *is a braided functor, A a bialgebra in* $\mathcal{C}$.

(i) The map $\Psi : \mathrm{Hom}_{\mathcal{C}}(A \otimes A, \underline{1}) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(A) \otimes \mathcal{F}(A), \underline{I})$ *sending* τ *to*

$$\tau_{\mathcal{F}} : \mathcal{F}(A) \otimes \mathcal{F}(A) \xrightarrow{\varphi_{2,A,A}} \mathcal{F}(A \otimes A) \xrightarrow{\mathcal{F}(\tau)} \mathcal{F}(\underline{1}) \xrightarrow{\varphi_0^{-1}} \underline{I},$$

is a morphism of monoids.

If $\tau : A \otimes A \to \underline{1}$ *is a* 2*-cocycle of A then* $\tau_{\mathcal{F}}$ *is a* 2*-cocycle of* $\mathcal{F}(A)$.

(ii) If $\mathcal{F}$ *is, moreover, a braided equivalence then* Ψ *is an isomorphism of monoids, so any* 2*-cocycle of* $\mathcal{F}(A)$ *equals* $\tau_{\mathcal{F}}$ *for a certain* 2*-cocycle* τ *of A in* $\mathcal{C}$. *Furthermore,* τ *is invertible if and only if so is* $\tau_{\mathcal{F}}$, *and* $\mathcal{F}(A_\tau) = \mathcal{F}(A)_{\tau_{\mathcal{F}}}$ *as bialgebras in* $\mathcal{D}$.

The ideal case

Theorem
Let $A \in {}^H_H\mathcal{YD}$ be a bialgebra (resp. Hopf algebra) and $\vartheta : \mathcal{F}_l(A) \otimes_H \mathcal{F}_l(A) \to H$ an invertible 2-cocycle of $\mathcal{F}_l(A) := A \otimes H$ in ${}^H_H\mathcal{M}^H_H$. Then there exists an invertible 2-cocycle $\widetilde{\vartheta}$ of A in ${}^H_H\mathcal{YD}$ such that $\mathcal{F}_l(A)_\vartheta = \mathcal{F}_l(A_{\widetilde{\vartheta}})$ as bialgebras (resp. Hopf algebras) in ${}^H_H\mathcal{M}^H_H$. Consequently, if $(A \times H)_\vartheta$ is the quasi-bialgebra (resp. quasi-Hopf algebra) corresponding to the bialgebra (resp. Hopf algebra) $\mathcal{F}_l(A)_\vartheta$ in ${}^H_H\mathcal{M}^H_H$ then $(A \times H)_\vartheta = A_{\widetilde{\vartheta}} \times H$ as quasi-bialgebras (resp. quasi-Hopf algebras).

- When $K = R \times H$, one can work over H; K is a bimonoid in ${}_H\mathcal{M}_H$, and for such a context a theory of 2-cocycles and deformations produced by them exists.

The bimonoid case

- If $i : H \to K$ is a quasi-Hopf algebra morphism, K is an algebra in ${}_H\mathcal{M}_H$ via m_K and i.
- A normalized 2-cocycle of K is an H-bilinear morphism $\omega : K \otimes_H K \to k$ s.t. $\omega(1_K \otimes_H x){=}\omega(x \otimes_H 1_K){=}\varepsilon_K(x)$,

$$\omega(x_1 \otimes_H y_1)\omega(x_2y_2 \otimes_H z){=}\omega(y_1 \otimes_H z_1)\omega(x \otimes_H y_2z_2).$$

- For $K = R \times H$, owing to [2], giving an (invertible) normalized 2-cocycle σ on $R \times H$ is equivalent to giving an (invertible) normalized left H-linear morphism $\vartheta : R\widetilde{\otimes}R \to k$ obeying

$$\vartheta((s\widetilde{\otimes}\mathfrak{t})_{\underline{1}})\vartheta(r\widetilde{\otimes}\underline{m}_A((s\widetilde{\otimes}\mathfrak{t})_{\underline{2}}) = \vartheta((x^1 \cdot r\widetilde{\otimes}x^2 \cdot s)_{\underline{1}})\vartheta(\underline{m}_A((x^1 \cdot r\widetilde{\otimes}x^2 \cdot s)_{\underline{1}})\widetilde{\otimes}\mathfrak{t}).$$

- $R\widetilde{\otimes}R$ is $R \otimes R$ with the braided monoidal algebra, coalgebra structure given by the tensor product of R and itself in ${}_H^H\mathcal{YD}$.

[2]Bulacu, Popescu, T., Double wreath quasi-Hopf algebras, J. Algebra (2025)

- We consider almost (invertible) normalized 2-cocycles on $C\widetilde{\otimes}\overline{B}$ in ${}^H_H\mathcal{YD}$ of the form $\vartheta = \underline{\varepsilon}_C \otimes \Sigma \otimes \underline{\varepsilon}_B$ for a suitable k-linear $\Sigma : \overline{B}\widetilde{\otimes}C \to k$.
- We replace $\overline{B}$ by an arbitrary bialgebra A in ${}^H_H\mathcal{YD}$ such that the tensor product algebra and coalgebra structures afford on $C \otimes A$ a braided bialgebra structure, denoted by $C\widetilde{\otimes}A$.

Theorem

ϑ is an almost (invertible) normalized 2-cocycle iff $\Sigma : A \otimes C \to k$ is left H-linear (convolution invertible in ${}_H\mathcal{M}$), and

$$\Sigma(1_A\widetilde{\otimes}c) = \underline{\varepsilon}_C(c)\ ,\ \Sigma(a\widetilde{\otimes}1_C) = \underline{\varepsilon}_A(a);$$

$$\Sigma(aa'\widetilde{\otimes}c) = \Sigma(X^1 \cdot a\widetilde{\otimes}x^3X^3_2 \cdot c_{\underline{2}})\Sigma(x^1X^2 \cdot a'\widetilde{\otimes}x^2X^3_1 \cdot c_{\underline{1}});$$

$$\Sigma(a \otimes cc') = \Sigma(y^1X^1x^1_1 \cdot a_{\underline{1}}\widetilde{\otimes}y^2(X^2x^1_2 \cdot a_{\underline{2}})_{[-1]}X^3x^2 \cdot c)$$
$$\Sigma(y^3 \cdot (X^2x^1_2 \cdot a_{\underline{2}})_{[0]}\widetilde{\otimes}x^3 \cdot c').$$

2-cocycles for double biproducts

When we take $A = \overline{B}$ as bialgebra in ${}_H^H\mathcal{YD}$, keeping in mind the bialgebra structure of $\overline{B}$, an almost dual skew pairing between $\overline{B}$, C is a k-linear morphism $\Sigma : B \otimes C \to k$ satisfying

$$\Sigma(b \cdot S^{-1}(h_1) \otimes h_2 \cdot c) = \varepsilon(h)\Sigma(b \otimes c),$$

$$\Sigma(1_B \otimes c) = \underline{\varepsilon}_C(c), \Sigma(b \otimes 1_C) = \underline{\varepsilon}_B(b);$$

$$\Sigma(b_{(0)}(b' \cdot b_{(1)}), c) = \Sigma(b \cdot S^{-1}(X^1 g^1) \otimes x^3 X^3_2 \cdot c_{\underline{2}})$$

$$\Sigma(b' \cdot S^{-1}(x^1 X^2 g^2) \otimes x^2 X^3_1 \cdot c_{\underline{1}});$$

$$\Sigma(b \otimes cc') = \Sigma\big((b_{\underline{1}})_{(\underline{0})} \cdot Y^2 p^1_2 S^{-1}(y^1 X^1 x^1_1 G^1) \otimes$$

$$y^2 g^1 S((b_{\underline{2}} \cdot S^{-1}(f^1 X^2 x^1_2 G^2 S(Y^1 p^1_1)(b_{\underline{1}})_{(\underline{1})} Y^3 p^2))_{(1)}) f^2 X^3 x^2 \cdot c\big)$$

$$\Sigma\big((b_{\underline{2}} \cdot S^{-1}(f^1 X^2 x^1_2 G^2 S(Y^1 p^1_1)(b_{\underline{1}})_{(\underline{1})} Y^3 p^2))_{(0)} \cdot S^{-1}(y^3 g^2)$$

$$\otimes x^3 \cdot c'\big).$$

2-cocycles for double biproducts

- Moving backwards, $\Sigma : B \otimes C \to k$ defines ϑ that defines ω, the later being a normalized invertible 2-cocycle on $(C\widetilde{\otimes}\overline{B}) \times H$ over H, Explicitly, $\omega : ((C\widetilde{\otimes}\overline{B}) \times H) \otimes_H ((C\widetilde{\otimes}\overline{B}) \times H) \to k$ is given by

$$\begin{aligned}\omega((c\widetilde{\otimes}b) \times h \otimes_H (c'\widetilde{\otimes}b') \times h') &= \varepsilon(h')\vartheta(c\widetilde{\otimes}b \otimes h \cdot (c'\widetilde{\otimes}b')) \\ &= \varepsilon(h')\underline{\varepsilon}_B(b')\Sigma(b \otimes h \cdot c'),\end{aligned}$$

  for all $b' \in B$, $c, c' \in C$ and $h, h' \in H$.
- At a first sight is quite impossible to find such a Σ. But, using the quasi-Hopf algebra isomorphism $\chi : C \times H \times B \to (C\widetilde{\otimes}\overline{B}) \times H$ one can see that the Σ's are in a one to one correspondence to certain H-balanced morphisms $\sigma : B \otimes A \to k$, morphisms that can be determined much more easily.

Theorem
Let H be a quasi-Hopf algebra, $C \in {}_H^H\mathcal{YD}$ and $B \in \mathcal{YD}_H^H$ braided Hopf algebras, and $\overline{B}$ the braided Hopf algebra in ${}_H^H\mathcal{YD}$ associated to B. Then there is a one to one correspondence between:

(i) almost (invertible) dual skew parings $\Sigma : \overline{B} \otimes C \to k$, and

(ii) H-balanced morphisms $\sigma : B \otimes C \to k$ satisfying

$$\sigma(b \otimes cc') = \sigma(b_{\underline{1}} \otimes a)\sigma(b_{\underline{2}} \otimes a'),$$
$$\sigma(bb' \otimes c) = \sigma(b \otimes b'_{(1)} \cdot c_{\underline{2}})\sigma(b'_{(0)} \otimes c'_{\underline{1}}).$$

$(C \times H \times B)^{\widehat{\sigma}}$ and $((C\widetilde{\otimes}\overline{B}) \times H)^{\omega}$ are isomorphic quasi-Hopf algebras.

Double biproduct quasi-Hopf algebras of dimension 32

- $H_\pm(8)$ are the quasi-Hopf algebras introduced in [3].
- As k-algebras, $H_\pm(8)$ are unital, generated by g, x with relations $g^2 = 1$, $x^4 = 0$ and $gx = -gx$.
- The (non-coassociative) coalgebra structures of $H_\pm(8)$ are given by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1,$$
$$\Delta(x) = x \otimes (p_+ \pm ip_-) + 1 \otimes p_+x + g \otimes p_-x, \quad \varepsilon(x) = 0,$$

extended as algebra morphisms, where $p_\pm = \frac{1}{2}(1 \pm g)$.
- $\{g^a x^b \mid 0 \leq a \leq 1, 0 \leq b \leq 3\}$ is a common basis for $H_\pm(8)$, two quasi-Hopf algebras with reassociator $\Phi = 1 \otimes 1 \otimes 1 - 2p_- \otimes p_- \otimes p_-$ and antipode defined by $S(g) = g$, $S(x) = -x(p_+ \pm ip_-)$, extended as an anti-algebra morphism, and distinguished elements $\alpha = g$ and $\beta = 1$.

[3] P. Etingof, S. Gelaki, Finite dimensional quasi-Hopf algebras with radical of codimension 2, Math. Res. Lett. **11** (2004), 685–696.

Double biproduct quasi-Hopf algebras of dimension 32

- $H_{\pm}(8)$ contain $H(2)$ as a quasi-Hopf subalgebra.
- $H(2)$ is the group algebra of k and the cyclic group $\langle g \rangle$, a 2-dimensional quasi-Hopf algebra with coalgebra structure, reassociator Φ and antipode (S, α, β) given by the same relations as in the case of $H_{\pm}(8)$.
- The biproduct quasi-Hopf algebras that identify to $H_{\pm}(8)$ as quasi-Hopf algebras are defined by the following braided Hopf algebras $R_{\pm} \in {}_{H(2)}^{H(2)}\mathcal{YD}$.
- As vector spaces, $R_{\pm}$ are generated by 1, $u_{\pm} := (p_{-} \pm ip_{+})x$, $v := gx^2$ and $w_{\pm} := (p_{-} \mp ip_{+})x^3$, and are Yetter-Drinfeld modules over $H(2)$ with structures defined by

$$g \triangleright 1 = 1,\ g \triangleright u_{\pm} = -u_{\pm},\ g \triangleright v = v,\ g \triangleright w_{\pm} = -w_{\pm};$$

$$1 \mapsto 1 \otimes 1,\ u_{\pm} \mapsto (p_{+} \pm ip_{-}) \otimes u_{\pm},\ v \mapsto 1 \otimes v \text{ and}$$

$$w_{\pm} \mapsto (p_{+} \pm ip_{-}) \otimes w_{\pm}.$$

Double biproduct quasi-Hopf algebras of dimension 32

- $R_\pm$ are unital braided algebras with unit 1 and multiplication $\bullet$ determined by

$$u_\pm \bullet u_\pm = \mp iv,\ u_\pm \bullet v = w_\pm,\ v \bullet u_\pm = -w_\pm,$$
$$u_\pm \bullet w_\pm = v \bullet v = v \bullet w_\pm = w_\pm \bullet u_\pm = w_\pm \bullet v = w_\pm \bullet w_\pm = 0,$$

- and counital braided coalgebras with counits $\underline{\varepsilon}_\pm$ and comultiplications $\underline{\Delta}_\pm$ given by $\underline{\varepsilon}_\pm(1) = 1$, $\underline{\varepsilon}_\pm(u_\pm) = \underline{\varepsilon}_\pm(v) = \underline{\varepsilon}_\pm(w_\pm) = 0$,

$$\underline{\Delta}_\pm(u_\pm) = 1 \otimes u_\pm + u_\pm \otimes 1,$$
$$\underline{\Delta}_\pm(v) = v \otimes 1 + 1 \otimes v - \omega_\mp u_\pm \otimes u_\pm,\ \text{where } \omega_\mp := 1 \mp i, \text{ and}$$
$$\underline{\Delta}_\pm(w_\pm) = w_\pm \otimes 1 + 1 \otimes w_\pm \pm iu_\pm \otimes v \mp iv \otimes u_\pm.$$

- Finally, the braided antipode $\underline{S}_\pm$ of $R_\pm$ is characterized by

$$\underline{S}_\pm(1) = 1,\ \underline{S}_\pm(u_\pm) = -u_\pm,\ \underline{S}_\pm(v) = \pm iv,\ \underline{S}_\pm(w_\pm) = \pm iw_\pm.$$

Double biproduct quasi-Hopf algebras of dimension 32

- For Take $C = R_+$ and $\overline{B} = R_-$, by using the structures of $C, \overline{B}$ in ${}_{H(2)}^{H(2)}\mathcal{YD}$, one can check easily that $c_{R_+,R_-} \circ c_{R_-,R_+} = \mathrm{Id}_{R_- \otimes R_+}$ (c is the braiding of ${}_{H(2)}^{H(2)}\mathcal{YD}$).
- Thus $R := R_+ \widetilde{\otimes} R_-$ is a braided Hopf algebra in ${}_{H(2)}^{H(2)}\mathcal{YD}$ and $R \times H(2)$, a 32-dimensional quasi-Hopf algebra, identifies to a double biproduct quasi-Hopf algebra.
- The 2-cocycles of $R \times H$ defined by an almost dual pairing Σ between R_- and R_+ are parametrized by $a \in k$, since the only non-zero values of Σ are $\Sigma(1 \otimes 1) = 1$, $\Sigma(u_- \otimes u_+) = a$, $\Sigma(v \otimes v) = -\omega_+ a^2$ and $\Sigma(w_- \otimes w_+) = -\omega_- a^3$.
- Having Σ, we have a 2-cocycle ω on $R \times H$, and therefore we can apply the bosonization process to $R \times H$ and ω.

Quasi free (left) Yetter-Drinfeld datum

Let H be a quasi-Hopf algebra with bijective antipode.

Definition

A quasi free (left) Yetter-Drinfeld datum over H (free YD-datum for short) is a triple $((e_i)_{i\in I}, (\chi_i)_{i\in I}, R)$ consisting of a family of elements $(e_i)_{i\in I}$ indexed by a non-empty set I, a family of characters $(\chi)_{i\in I}$ of H indexed by I and an element $R \in H \otimes H$ satisfying $(\mathrm{Id}_H \otimes \varepsilon)(R) = 1$ and

$$\begin{aligned}(\mathrm{Id} \otimes \Delta)(R) &= (\Phi_{231})^{-1} R_{13} \Phi_{213} R_{12} (\Phi_{123})^{-1} \\ &= x^3 R^1 X^2 r^1 y^1 \otimes x^1 X^1 r^2 y^2 \otimes x^2 R^2 X^3 y^3, \qquad (1) \\ \Delta^{op}(h) R &= R \Delta(h) \\ i.e. \quad h_2 R^1 \otimes h_1 R^2 &= R^1 h_1 \otimes R^2 h_2, \ \forall\ h \in H, \qquad (2)\end{aligned}$$

where $R = R^1 \otimes R^2 = r^1 \otimes r^2$ are two copies of R.

• Note that the three conditions imposed to the above $R \in H \otimes H$ are part of the definition of an R-matrix for H. Thus, a couple (H, R) with $R \in H \otimes H$ obeying $\varepsilon(R^2)R^1 = 1$, (0.1) and (0.2) will be called in what follows a (left) semi-quasitriangular quasi-Hopf algebra (semi-QT for short). Also, we say that R is a (left) semi R-matrix for H.

• A semi R-matrix for H is always invertible, provided that S is bijective. As in the quasitriangular case, one can see that the inverse of R is given by

$$R^{-1} := \tilde{q}^2 y_2^2 R^1 p^1 \otimes y^3 S^{-1}(\tilde{q}^1 y_1^2 R^2 p^2) y^1 \tag{3}$$

$$= \tilde{q}_1^2 X^1 R^1 p^1 \otimes \tilde{q}_2^2 X^3 S^{-1}(\tilde{q}^1 X^1 R^2 p^2). \tag{4}$$

Lemma

Giving a left Yetter-Drinfeld module structure on a one dimensional vector space is equivalent to giving a pair $(\chi, \mathfrak{K})$ *consisting of a character* χ *of* H *and an element* $\mathfrak{K} \in H$ *such that* $\varepsilon(\mathfrak{K}) = 1$, $\chi(h_2)h_1\mathfrak{K} = \chi(h_1)\mathfrak{K}h_2$, *for all* $h \in H$, *and*

$$\Delta(\mathfrak{K}) = \chi(x^3X^2y^1)x^1X^1\mathfrak{K}y^2 \otimes x^2\mathfrak{K}X^3y^3. \tag{5}$$

Corollary

If H *possess a semi* R*-matrix* R, *any character* χ *of* H *determines a left Yetter-Drinfeld module structure on each one dimensional vector space.*

Denote by $\mathcal{YD}_1$ the set of pairs $(\mathfrak{K}, \chi)$ consisting of an element $\mathfrak{K} \in H$ and a character χ of H such that, for all $h \in H$,

$$\begin{aligned} \Delta(\mathfrak{K}) &= \chi(x^3 X^2 y^1) x^1 X^1 \mathfrak{K} y^2 \otimes x^2 \mathfrak{K} X^3 y^3 \\ \chi(h_2) h_1 \mathfrak{K} &= \chi(h_1) \mathfrak{K} h_2 \\ \varepsilon(\mathfrak{K}) &= 1 \end{aligned} \tag{6}$$

Lemma

For $(\mathfrak{K}_1, \chi_1), (\mathfrak{K}_2, \chi_2) \in \mathcal{YD}_1$*, define*

$$(\mathfrak{K}_1, \chi_1) * (\mathfrak{K}_2, \chi_2) := (\chi_1(X^2 x^1 Y^1)\chi_2(X^3 x^3 Y^2) X^1 \mathfrak{K}_1 x^2 \mathfrak{K}_2 Y^3, \chi_1\chi_2).$$

Then, the following assertions hold:

(i) The operation $*$ *is an associative product in* $\mathcal{YD}_1$*;*

(ii) $(1, \varepsilon)$ *is a neutral element of* $\mathcal{YD}_1$*, and with respect with it any element* $(\mathfrak{K}, \chi)$ *is invertible, with inverse given by* $(\mathfrak{K}^{-1}, \chi^{-1})$*, where* χ^{-1} *is the (convolution) inverse of* χ *and*

$$\mathfrak{K}^{-1} := \chi(f^2 g^1) S^{-1}(f^1 \mathfrak{K} g^2), \tag{7}$$

where $f = f^1 \otimes f^2$ *is the Drinfeld twist and* $g = g^1 \otimes g^2$ *is its inverse;*

(iii) $(\mathcal{YD}_1, *)$ *is a commutative group.*

• Let $\mathcal{E} = (e_i)_i$ be a family of elements indexed by an non-empty set I and $(\chi_i)_{i\in I}$ a family of characters of H.

• A (left) quasi-word in alphabet $\mathcal{E}$ is a sequence

$$w = w(i_1, \cdots, i_n) := e_{i_1}(e_{i_2}(e_{i_3}\cdots(e_{i_{n-1}}e_{i_n})\cdots))$$

with n a non-zero natural number (called in what follows the length of w) and $i_1, \cdots, i_n \in I$; we include also the empty word $\emptyset$.

• The presence of the parenthesis is justified by the fact that the algebra we want to built might be non-associative, as for an algebra in ${}^H_H\mathcal{YD}$ the associativity of the multiplication is controlled by the associativity constraint of ${}^H_H\mathcal{YD}$, and thus by the reassociator Φ.

We define the (left) quasi-free k-algebra on the set $\mathcal{E}$, denoted by $k\{(\mathcal{E}\}$, as being the k-vector space with basis the all (left) quasi-words in alphabet $\mathcal{E}$, including the empty word $\emptyset$; the multiplication between two non-empty quasi-words w and $w' = w(i'_1, \cdots, i'_m) = e_{i'_1}(e_{i'_2}(e_{i'_3} \cdots (e_{i'_{m-1}} e_{i'_m}) \cdots))$ is a scalar multiple of the "concatenation" of the two quasi-words,

$$ww' = \kappa w(i_1, \cdots, i_n, i'_1, \cdots, i'_m) = \kappa e_{i_1}(e_{i_2} \cdots (e_{i_{n-1}}(e_{i_n}(e_{i'_1}(\cdots (e_{i'_{m-1}} e_{i'_m}) \cdot$$

with the scalar κ determined by the following rule:

$$(e_{i_1} e_{i_2}) e_{i_3} = \chi_{i_1}(X^1)\chi_{i_2}(X^2)\chi_{i_3}(X^3) e_{i_1}(e_{i_2} e_{i_3}), \; \forall \; i_1, i_2, i_3 \in I, \quad (8)$$

extended to arbitrary non-empty quasi-words by considering $\chi_{w(i_1,\cdots,i_n)} := \chi_{i_1}(\chi_{i_2} \cdots (\chi_{i_{n-1}} \chi_{i_n}) \cdots)$; more generally, if the order of the parenthesis in a concatenation is not the standard one then we adapt the definition of χ associated to concatenation accordingly. The unit is the empty word.

Proposition

Let H be a quasi-Hopf algebra, $\mathcal{E} = (e_i)_{i\in I}$ a family of elements and $(\mathfrak{K}_i, \chi_i)_{i\in I}$ a family with elements in $\mathcal{YD}_1$. Then $k\{(\mathcal{E}\}$ admits a unique algebra structure in ${}_H^H\mathcal{YD}$ such that, for all $h \in H$ and $i \in I$,

$$h \cdot e_i = \chi_i(h)e_i \text{ and } \lambda(e_i) = \mathfrak{K}_i \otimes e_i, \tag{9}$$

where λ is the left coaction of H on $k\{(\mathcal{E}\}$.

Lemma

Let $\iota : \mathcal{E} \hookrightarrow k\{(\mathcal{E}\}$ be the inclusion map and A an algebra in ${}_H^H\mathcal{YD}$. Then, for any map $f : \mathcal{E} \to A$ obeying, for all $h \in H$ and $i \in I$,

$$\chi_i(h)f(e_i) = h \cdot f(e_i) \text{ and } \mathfrak{K}_i \otimes f(e_i) = f(e_i)_{[-1]} \otimes f(e_i)_{[0]}, \tag{10}$$

there exists a unique morphism $\overline{f} : k\{(\mathcal{E}\} \to A$ of algebras in ${}_H^H\mathcal{YD}$ such that $\overline{f}\iota = f$.

• For any $1 \leq s \leq n$, denote by $S_{s,n-s}$ the set of $(s, n-s)$-shuffles, that is the set of permutations $\sigma \in S_n$ for which $\sigma(1) < \cdots < \sigma(s)$ and $\sigma(s+1) < \cdots < \sigma(n)$.

• It is well-known that $S_{s,n-k}$ has $\binom{n}{s}$ elements, so S_n contains in total 2^n shuffles; we included also $S_{0,n} := \{e\} = S_{n,0}$, e being the identical permutation of S_n.

• For $\sigma \in S_n$, we denote by $\mathrm{Inv}(\sigma)$ the set of inversions of σ; by convention, if $(u, v) \in \mathrm{Inv}(\sigma)$ then $u < v$, and so $\sigma(u) > \sigma(v)$.

Proposition

There is a unique coalgebra structure $(\underline{\Delta}, \underline{\varepsilon})$ *on* $k\{(\mathcal{E}\}$ *in* ${}^H_H\mathcal{YD}$ *such that the comultiplication* $\underline{\Delta}$ *is a morphism of algebras in* ${}^H_H\mathcal{YD}$ *and* $\underline{\Delta}(e_i) = e_i \otimes \mathbf{1} + \mathbf{1} \otimes e_i$*, for all* $i \in I$*. Furthermore, for any quasi-word* $w = w(i_1, \cdots, i_n)$ *we have*

$$\underline{\Delta}(w) = \sum_{s=0}^{n} \sum_{\sigma \in S_{s,n-s}} \prod_{u=1}^{n} \left(\prod_{(u,v) \in \mathrm{Inv}(\sigma^{-1})} \chi_{i_v} \right) \left(\mathfrak{K}_{i_u}\right) w\big(i_{\sigma(1)}, \cdots, i_{\sigma(s)}\big) \otimes w\big(i_{\sigma(s+1)}, \cdots, i_{\sigma(n)}\big), \tag{11}$$

and the counit $\underline{\varepsilon}$ *is a morphism of algebras in* ${}^H_H\mathcal{YD}$ *determined by* $\underline{\varepsilon}(e_i) = 0$*, for all* $i \in I$*. Consequently,* $k\{(\mathcal{E}\}$ *is a bialgebra in* ${}^H_H\mathcal{YD}$*.*

Theorem

Let $\mathcal{E} = (e_i)_{i\in I}$ be a family of elements and $(\mathfrak{K}_i, \chi_i)_{i\in I}$ a family of elements of $\mathcal{YD}_1$. Then, the (left) quasi-free algebra on the set $\mathcal{E}$, $k\{(\mathcal{E}\}$ is a braided Hopf algebra in ${}_H^H\mathcal{YD}$ with the following structure:

- *$k\{(\mathcal{E}\}$ is a left Yetter-Drinfeld module with H-action defined by $h \cdot \mathbf{1} = \varepsilon(h)\mathbf{1}$ and $h \cdot w(i_1, \cdot, i_n) = \chi_w(h)w(i_1, \cdots, i_n)$, for all $h \in H$ and non-empty quasi-word $w(i_1, \cdots, i_n)$, and H-coaction determined by $\mathbf{1} \mapsto 1 \otimes \mathbf{1}$ and $w = w(i_1, \cdots, i_n) \mapsto \mathfrak{K}_w \otimes w$;*
- *the multiplication $\underline{m}$ of $k\{(\mathcal{E}\}$ is given by (8) and the unit is the empty word $\mathbf{1}$;*
- *the comultiplication $\underline{\Delta}$ of $k\{(\mathcal{E}\}$ is defined by $\underline{\Delta}(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$ and (11), while the counit $\underline{\varepsilon}$ of $k\{(\mathcal{E}\}$ is defined by $\underline{\varepsilon}(\mathbf{1}) = 1$ and $\underline{\varepsilon}(w) = 0$, for any non-empty quasi-word w;*

Theorem (Continued)

• *the braided antipode* $\underline{S}$ *of* $k\{(\mathcal{E}\}$ *is determined by* $\underline{S}(\mathbf{1}) = \mathbf{1}$, $\underline{S}(e_i) = -e_i$, $\underline{S}(e_{i_1}e_{i_2}) = \chi_{i_2}(\mathfrak{K}_1)e_{i_2}e_{i_1}$ *and*

$$\underline{S}(w) = (-1)^n \Big(\prod_{j=2}^{n} \chi_{i_j} \cdots \chi_{i_n}(\mathfrak{K}_{i_{j-1}}) \Big) \Big(\prod_{j=3}^{n} \chi_{i_j}(X^1)\chi_{i_{j-1}} \cdots \chi_{i_2}(X^2)\chi_{i_1}(X^3) \Big) w(i_n, \cdots, i_1) \tag{12}$$

for any non-empty quasi-word $w = w(i_1, \cdots, i_n)$ *of length* $n \geq 3$.

Corollary
Let H and $\{(\mathfrak{K}_i, \chi_i)\}_{i\in I}$ as previously. Given a family of symbols $\mathcal{E} = (E_i)_{i\in I}$, the biproduct quasi-Hopf algebra $\mathfrak{e}$, associated to the braided Hopf algebra $k\{(\mathcal{E}\}$, admits the following presentation:

- **Algebra Structure:** *As a unital associative algebra, $\mathfrak{e}$ is generated by the family $\{E_i\}_{i\in I}$ and H, subject to the relations:*

$$hE_i = \chi_i(h_1)E_i h_2$$

for all $i \in I$ and $h \in H$.

- **Coalgebra Structure:** *The comultiplication $\Delta_{\mathfrak{e}}$ and the counit $\varepsilon_{\mathfrak{e}}$ are determined by:*

$$\Delta_{\mathfrak{e}}(E_i) = \chi_i(x^1)E_i x^2 \otimes x^3 + \chi_i(X^2 x^1)X^1 \mathfrak{K}_i x^2 \otimes E_i X^3 x^3,$$
$$\Delta_{\mathfrak{e}}(h) = \Delta(h),$$

Corollary (continued)
and

$$\varepsilon_{\mathfrak{e}}(E_i) = 0, \quad \varepsilon_{\mathfrak{e}}(h) = \varepsilon(h),$$

for all $i \in I$ *and* $h \in H$*. These maps are extended to all of* $\mathfrak{e}$ *as algebra homomorphisms.*

- **Antipode:** *The antipode* $S_{\mathfrak{e}}$ *is defined by:*

  $$S_{\mathfrak{e}}(E_i) = -\chi_i(X^1 p_2^1) S(X^1 p_1^1 \mathfrak{K}_i) \alpha E_i X^3 p^2 \quad \text{and} \quad S_{\mathfrak{e}}(h) = S(h),$$

  for all $i \in I$ *and* $h \in H$*, extended as an algebra anti-homomorphism.*
- **Quasi-Hopf Structure:** *The Drinfeld associator and the distinguished elements* α, β *coincide with those of* H*.*

The right version of the quasi free algebra

Consider $\mathcal{F} = (f_j)_{j\in J}$ a family of elements indexed by a non-empty set J and $(\overline{\chi}_j)_{j\in J}$ a family of characters of H. For any $j \in J$, kf_j as a right H-module via the action given by $\overline{\chi}_j$: $f_j \cdot h = \overline{\chi}_j(h)f_j$, for all $h \in H$. Then, kf_j is, moreover, a right Yetter-Drinfeld module over H if and only if there exists an element $\mathfrak{G}_i$ in H such that,

$$\begin{aligned} \Delta(\mathfrak{G}_j) = \overline{\chi}_j(y^3X^2x^1)y^1X^1\mathfrak{G}_jx^2 \otimes y^2\mathfrak{G}_jX^3x^3, \\ \overline{\chi}_j(h_1)\mathfrak{G}_jh_2 = \overline{\chi}_j(h_2)h_1\mathfrak{G}_j, \\ \varepsilon(\mathfrak{G}_j) = 1, \end{aligned} \tag{13}$$

for all $h \in H$. Denote by $\mathcal{YD}'_1$ the set of couples $(\mathfrak{G}, \overline{\chi})$ with $\overline{\chi}$ a character of H and $\mathfrak{G}$ an element of H satisfying (13). $\mathcal{YD}'_1$ is a commutative group under the law of composition $(\mathfrak{G}_1, \overline{\chi}_1)(\mathfrak{G}_2, \overline{\chi}_2) = (\mathfrak{G}_1 \diamond \mathfrak{G}_2, \overline{\chi}_1\overline{\chi}_2)$, where

$$\mathfrak{G}_1 \diamond \mathfrak{G}_2 := \chi_1(X^2x^1Y^1)\chi_2(X^3x^3Y^2)X^1\mathfrak{G}_1x^2\mathfrak{G}_2Y^3.$$

A right quasi-word in alphabet $\mathcal{F}$ is a sequence $v = v(j_1, \cdots, j_n) := ((\cdots((f_{j_1} f_{j_2}) f_{j_3}) \cdots) f_{j_{n-1}}) f_{j_n}$ with n a non-zero natural number (called the length of v) and $j_1, \cdots, j_n \in J$; we include also the empty word $\emptyset$.

We define the right quasi-free k-algebra on the set $\mathcal{F}$, denoted by $k\{\mathcal{F})\}$, as being the k-vector space with basis the all right quasi-words in alphabet $\mathcal{F}$, including the empty word $\emptyset$; the multiplication between v and $v' = v(j'_1, \cdots, j'_m) = (\cdots((f_{j'_1} f_{j'_2}) f_{j'_3}) \cdots f_{j'_{m-1}}) f_{j'_m}$ is a scalar multiple of the "concatenation" of the two quasi-words,

$$vv' = \kappa' v(j_1, \cdots, j_n, j'_1, \cdots, j'_m) = \kappa'(\cdots(((\cdots(f_{j_1} f_{j_2}) \cdots f_{j_{n-1}}) f_{j_n}) f_{j'_1}) \cdots f_{j'_{m-1}}) f_{j'_m}, \quad (14)$$

with the scalar κ' determined by the following rule:

$$(f_{j_1} f_{j_2}) f_{j_3} = \overline{\chi}_{j_1}(x^1) \overline{\chi}_{j_2}(x^2) \overline{\chi}_{j_3}(x^3) f_{j_1}(f_{j_2} f_{j_3}), \ \forall\ j_1, j_2, j_3 \in J. \quad (15)$$

To perform the **double biproduct** we need the compatibility relation, if $C = \mathfrak{e}$ is the Hopf algebra in ${}^H_H\mathcal{YD}$ and $B = \mathfrak{f}$ is the Hopf algebra in $\mathcal{YD}^H_H$ then for all $b \in B$ and $c \in C$,

$$b \otimes c = b_{(0)} \cdot c_{[-1]} \otimes b_{(1)} \cdot c_{[0]}.$$

Thus, for our structures and $b = v$, $c = w$, the above condition specializes as $v \otimes w = \overline{\chi}_v(\mathfrak{K}_w)\chi_w(\mathfrak{G}_v) v \otimes w$. Hence, me must have $\overline{\chi}_v(\mathfrak{K}_w)\chi_w(\mathfrak{G}_v) = 1$, for all v and w. This is equivalent to $\overline{\chi}_j(\mathfrak{K}_i)\chi_i(\mathfrak{G}_j) = 1$, for all $(i, j) \in I \times J$.

This is satisfied working with $\mathfrak{K}_i$'s and the $\mathfrak{G}_j$'s defined by an R-matrix of H, since $\mathfrak{K}_w = \chi_w(R^1)R^2$ and $\mathfrak{G}_v = \overline{\chi}_v(\overline{R}^2)\overline{R}^1$, and therefore

$$\begin{aligned}\overline{\chi}_v(\mathfrak{K}_w)\chi_w(\mathfrak{G}_v) &= \chi_w(R^1)\overline{\chi}_v(R^2)\overline{\chi}_v(\overline{R}^2)\chi_w(\overline{R}^1)\\ &= \chi_w(R^1\overline{R}^1)\overline{\chi}_v(R^2\overline{R}^2) = 1,\end{aligned}$$

as needed.

Proposition
Suppose that $\mathfrak{e}$ and $\mathfrak{f}$ are compatible, in the sense that $\overline{\chi}_j(\mathfrak{K}_i)\chi_i(\mathfrak{G}_j) = 1$, for all $(i, j) \in I \times J$. Then, the double biproduct quasi-Hopf algebra of $\mathfrak{e}$ and $\mathfrak{f}$ over H, denoted by $DB_H(\mathfrak{e}, \mathfrak{f}) = \mathfrak{e} \times H \times \mathfrak{f}$, can be described as follows:

- *as an associative algebra, $DB_H(\mathfrak{e}, \mathfrak{f})$ is unital, generated by the elements E_i's, F_j's and H with relations*

$$hE_i = \chi_i(h_1)E_ih_2,\ F_jh = \overline{\chi}_j(h_2)h_1F_j,\ E_iF_j = F_j,$$

for all $(i, j) \in I \times J$ and $h \in H$;

- *the quasi-coalgebra structure of $DB_H(\mathfrak{e}, \mathfrak{f})$ is defined by*

$$\Delta(E_i) = \chi_i(x^1)E_ix^2 \otimes x^3 + \chi_i(X^2x^1)X^1\mathfrak{K}_ix^2 \otimes E_iX^3x^3,\ \varepsilon(E_i) = 0,$$
$$\Delta(F_j) = \overline{\chi}_j(x^3)x^1 \otimes x^2F_j + \overline{\chi}_j(x^3X^2)x^1F_j \otimes x^2\mathfrak{G}_jX^3,\ \varepsilon(F_j) = 0,$$

for all $(i, j) \in H$, and the restriction of Δ (resp. ε) to H equals the comultiplication of H (resp. the counit of H);

Proposition (Continued)
• *with the above strutures $DB_H(\mathfrak{e},\mathfrak{f})$ is a quasi-bialgebra with reassociator Φ and, moreover, a quasi-Hopf algebra with antipode $\mathcal{S}$ given by the distinguished elements α, β and*

$$\mathcal{S}(E_i) = -\chi_i(X^1 p_2^1) S(X^1 p_1^1 \mathfrak{K}_i) \alpha E_i X^3 p^2,$$
$$\mathcal{S}(F_j) = -\overline{\chi}_j(\tilde{q}_1^2 X^2) \tilde{q}^1 X^1 F_j \beta S(\mathfrak{G}_j \widetilde{q}_2^2 X^3),$$

for all $i \in I$ and $j \in J$, extended as an anti-morphism of algebras and such that its restriction to H equals S.

A double biproduct can be always identified, up to an isomorphism, with a left (or right) biproduct. The first step is to associate to $\mathfrak{f}$, a braided Hopf algebra $\bar{\mathfrak{f}}$ in ${}^H_H\mathcal{YD}$.

Proposition

$\bar{\mathfrak{f}} = \mathfrak{f}$ as a vector space, and a braided Hopf algebra in ${}^H_H\mathcal{YD}$ with structure given by:

- *$\bar{\mathfrak{f}}$ is a left H-module with action defined by $h \cdot f_j = \overline{\chi}_j^{-1}(h) f_j$, for all $h \in H$, extended to the whole space $\bar{\mathfrak{f}}$ by using $h \cdot (\mathfrak{b}\mathfrak{b}') = (h_1 \cdot \mathfrak{b})(h_2 \cdot \mathfrak{b}')$, and a left YD-module over H with coaction determined by $f_j \mapsto \mathfrak{G}_j^{-1} \otimes f_j$, for all $j \in J$, extended to the whole space $\bar{\mathfrak{f}}$ as an algebra morphism, by using $(\mathfrak{b}\mathfrak{b}')_{[-1]} \otimes (\mathfrak{b}\mathfrak{b}')_{[0]} = X^1(x^1 Y^1 \cdot \mathfrak{b})_{[-1]} x^2 (Y^2 \cdot \mathfrak{b}')_{[-1]} Y^3 \otimes (X^2 \cdot (x^1 Y^1 \cdot \mathfrak{b})_{[0]})(X^3 x^3 \cdot (Y^2 \cdot \mathfrak{b}')_{[0]})$;*
- *the multiplication of $\bar{\mathfrak{f}}$ is given by the multiplication $\mathfrak{f}$ as follows: $\overline{v}\overline{v'} = \overline{\chi}_v^{-1}(S(\mathfrak{G}_v) f^1) \overline{\chi}_{v'}^{-1}(f^2) v v'$, for all v and v', where $\overline{v}$ is v viewed in $\bar{\mathfrak{f}}$ instead of $\mathfrak{f}$ and similar for v';*

We denote by $\mathcal{A}_H(\mathfrak{e},\mathfrak{f})$ the space $\mathfrak{e}\otimes\mathfrak{f}$ endowed with the tensor product algebra and coalgebra structure of $\mathfrak{e}$ and $\bar{\mathfrak{f}}$ in ${}_H^H\mathcal{YD}$. As we assumed that the compatibility relation holds, $\mathcal{A}_H(\mathfrak{e},\mathfrak{f})$ is a braided Hopf algebra in ${}_H^H\mathcal{YD}$. The associated biproduct quasi-Hopf algebra $\mathcal{A}_H(\mathfrak{e},\mathfrak{f})\times H$ has the following structure:

Algebra structure

A an algebra is generated by the elements $(E_i)_{i\in I}$, $(\overline{F}_j)_{j\in J}$ and H subject to the relations $hE_i=\chi_i(h_1)E_ih_2$, $h\overline{F}_j=\overline{\chi}_j^{-1}(h_1)\overline{F}_jh_2$ and $\overline{F}_jE_i=\overline{\chi}_j^{-1}(X^2x^1)\chi_i(X^1x^2\mathfrak{G}_j^{-1})E_i\overline{F}_jX^3x^3$, for all $(i,j)\in I\times J$ and $h\in H$.

Comultiplication and counit

$$\Delta(E_i) = \chi_i(x^1)E_i x^2 \otimes x^3 + \chi_i(X^2x^1)X^1\mathfrak{K}_i x^2 \otimes E_i X^3 x^3,$$
$$\Delta(\overline{F}_j) = \overline{\chi}_j^{-1}(x^1)F_j x^2 \otimes x^3 + \overline{\chi}_j^{-1}(X^2x^1)Y^1\mathfrak{G}_j^{-1}x^2 \otimes F_j X^3 x^3,$$
$$\varepsilon(E_i) = 0, \varepsilon(\overline{F}_j) = 0,$$

for all $(i,j) \in I \times J$, and on H they reduce to the comultiplication and the counit of H;

Antipode

$$\underline{\mathcal{S}}(E_i) = -\chi_i(X^1p_2^1)S(X^1p_1^1\mathfrak{K}_i)\alpha E_i X^3 p^2,$$
$$\underline{\mathcal{S}}(\overline{F}_j) = -\overline{\chi}_j^{-1}(X^2p_2^1)S(X^1p_1^1\mathfrak{G}_j^{-1})\alpha \overline{F}_j X^3 p^2.$$

Serre's relations

Let (H, R) be a QT quasi-Hopf algebra and q a non-zero scalar.

Denote by $\mathfrak{I}_{\mathfrak{e}}$ (resp. $\mathfrak{I}_{\mathfrak{f}}$) the ideal of $\mathfrak{e}$ (resp. $\mathfrak{f}$) generated by

$$\begin{aligned}
&\{e_ie_j - e_je_i \mid i \neq j \text{ s.t. } \chi_i(R^1)\chi_j(R^2) = \chi_j(R^1)\chi_i(R^2) = 1\} \\
&\quad \cup \{e_i(e_ie_j) - [2]e_i(e_je_i) + \chi_i(x^1x^3)\chi_j(x^2)e_je_i^2 \mid i \neq j \text{ s.t. } \\
&\quad \chi_i(R^1R^2) = q^2,\ \chi_i(R^1)\chi_j(R^2) = \chi_j(R^1)\chi_i(R^2) = q^{-1}\} \\
&(\{f_if_j - f_jf_i \mid i \neq j \text{ s.t. } \overline{\chi}_i(\overline{R}^1)\overline{\chi}_j(\overline{R}^2) = \overline{\chi}_j(\overline{R}^1)\overline{\chi}_i(\overline{R}^2) = 1\} \\
&\quad \cup \{\overline{\chi}_i(x^1x^3)\overline{\chi}_j(x^2)f_i(f_if_j) - [2](f_if_j)f_i + (f_jf_i)f_i \mid i \neq j \text{ s.t. } \\
&\quad \chi_i(\overline{R}^1\overline{R}^2) = q^2, \overline{\chi}_i(\overline{R}^1)\overline{\chi}_j(\overline{R}^2) = \overline{\chi}_j(\overline{R}^1)\overline{\chi}_i(\overline{R}^2) = q^{-1}\}),
\end{aligned}$$

where $[2] = q + q^{-1}$.

• Then $\mathfrak{I}_{\mathfrak{e}}$ (resp. $\mathfrak{I}_{\mathfrak{f}}$) is a braided Hopf ideal in $\mathfrak{e}$ (resp. $\mathfrak{f}$), and we can consider the quotient braided Hopf algebra $\mathfrak{e}' = \frac{\mathfrak{e}}{\mathfrak{I}_{\mathfrak{e}}}$ (resp. $\mathfrak{f}' = \frac{\mathfrak{f}}{\mathfrak{I}_{\mathfrak{f}}}$).

• When we perform the double biproduct quasi-Hopf algebra $\mathfrak{e}' \times H \times \mathfrak{f}'$ we have for it a similar description as for $\mathfrak{e} \times H \times \mathfrak{f}$, with mention that the relations amoung the algebra generators are enriched with the two sets of Serre relations described above.

2-cocycle deformation

Let A (for us $\mathfrak{e}'$) be a Hopf algebra in ${}^H_H\mathcal{YD}$ and B (for us $\mathfrak{f}'$) a Hopf algebra in $\mathcal{YD}^H_H$. Two cocycles for $A \times H \times B$ are produced by linear maps $\sigma : B \otimes A \to k$ satisfying the usual unital conditions and

$$\sigma(b \cdot h \otimes a) = \sigma(b \otimes h \cdot a), \tag{16}$$

$$\sigma(b \otimes aa') = \sigma(b_{\underline{1}} \otimes a)\sigma(b_{\underline{2}} \otimes a'), \tag{17}$$

$$\sigma(bb' \otimes a) = \sigma(b \otimes b'_{(1)} \cdot a_{\underline{2}})\sigma(b'_{(0)} \otimes a_{\underline{1}}), \tag{18}$$

for all $h \in H$, $a, a' \in A$ and $b, b' \in B$. More exactly, $\widehat{\sigma} : (A \times H \times B) \otimes_H (A \times H \times B) \to k$ defined by $\widehat{\sigma}(a \times h \times b \otimes_H a' \times h' \times b') = \underline{\varepsilon}_A(a)\sigma(a \otimes h \cdot b)\varepsilon(h')\underline{\varepsilon}_B(b')$ is a 2-cocycle for $A \times H \times B$.

- For us, σ as above are defined by

$$\sigma(f_j \otimes e_i) = \delta_{\chi_i,\overline{\chi}_j} \varpi_{i.j},$$

- $(A \times B \times H \times B)^{\widehat{\sigma}}$ has the same quasi-coalgebra structure as $A \times H \times B$, but the algebra structure changes as follows ($i \in I$, $j \in J$, $h \in H$):

$$hE_i = \chi_i(h_1)E_i h_2 \tag{19}$$

$$hF_j = \overline{\chi}_j(h_2)h_1 F_j, \tag{20}$$

$$[F_j, E_i] = (\chi_i(\overline{R}^2)\overline{R}^1 - \chi_i(R^1)R^2)\delta_{\chi_i,\overline{\chi}_j} \varpi_{i,j}. \tag{21}$$

- The antipode changes accordingly.

Drinfeld-Jimbo quasi-quantum groups

• Let $(I, \cdot)$ be a Cartan datum. Here $(\mathbb{Z}[I], +)$ is the free abelian group with basis $\{i, i \in I\}$. The elements of $\mathbb{Z}[I]$ are denoted by $\{K_\nu, \nu \in \mathbb{Z}[I]\}$; then $K_\mu K_\nu = K_{\mu+\nu}$, so K_0 is the neutral element of $\mathbb{Z}[I]$, and $K_\mu^{-1} = K_{-\mu}$, for all $\mu, \nu \in \mathbb{Z}[I]$.

• We assume I to be finite just to have an R-matrix for the Hopf algebra of functions associated to $\mathbb{Z}[I]$, $k^{\mathbb{Z}[I]}$. Actually, it its well-known that $R_I = \sum_{\mu,\nu\in\mathbb{Z}[I]} q^{\mu\cdot\nu} P_\mu \otimes P_\nu$ is an R-matrix for $k^{\mathbb{Z}[I]}$, where $(P_\mu)_{\mu\in\mathbb{Z}[I]}$ is the basis of $k^{\mathbb{Z}[I]}$ dual to the basis $(K_\mu)_{\mu\in\mathbb{Z}[I]}$ of $k[\mathbb{Z}[I]]$.

• We don't have non-trivial abelian 3-cocycles for the group $\mathbb{Z}[I]$, and therefore no quasitriangular structures in the quasi-Hopf sense for $k^{\mathbb{Z}[I]}$. Therefore, we have to tensorize $k^{\mathbb{Z}[I]}$ with a quasitriangular quasi-Hopf algebra (H, R).

For (H, R) as above, set $\mathbb{H} = k^{\mathbb{Z}[I]} \otimes H$, a QT quasi-Hopf algebra with R-matrix $\mathcal{R}$ given by

$$\mathcal{R} := R_I^1 \otimes R^1 \otimes R_I^2 \otimes R^2.$$

We take two families of characters $(\chi_i)_{i \in I} \in \hat{H} = \mathrm{Alg}_k(H, k)$ and $(\overline{\chi}_i)_{i \in I} \in \hat{H}$

We extend them to two families of characters for $\mathbb{H}$, $({\chi'}_i)_{i \in I}$ and $(\overline{\chi}'_i)_{i \in I}$, defined by

$$\chi'_i(P_\mu \otimes h) = \delta_{i,\mu} \chi_i(h),$$

and respectively by

$$\overline{\chi}'_i(P_\mu \otimes h) = \delta_{i,\mu} \overline{\chi}_i(h),$$

for all $\mu \in \mathbb{Z}[I]$ and $h \in H$.

• Consider $\mathfrak{e}$ the quasi-free algebra on the set $(e_i)_{i\in I}$ and characters $(\chi'_i)_{i\in I}$. By Theorem 2.10, $\mathfrak{e}$ is a braided Hopf algebra in ${}^{\mathbb{H}}_{\mathbb{H}}\mathcal{YD}$.

• Analogously, consider $\mathfrak{f}$ the quasi-free algebra on the set $(f_i)_{i\in I}$ and characters $(\overline{\chi}'_i)_{i\in I}$; $\mathfrak{f}$ is a braided Hopf algebra in $\mathcal{YD}^{\mathbb{H}}_{\mathbb{H}}$.

• Now, we consider the deformed double biproduct $DB_H(\mathfrak{e}', \mathfrak{f}')_{\hat{\sigma}} = (\mathfrak{e}' \times \mathbb{H} \times \mathfrak{f}')^{\hat{\sigma}}$, where

$$\sigma : \mathfrak{f}' \otimes \mathfrak{e}' \to k$$

is defined by $\sigma(f_j \otimes e_i) = \delta_{\chi_i, \overline{\chi}_j} \varpi_{i,j}$; here $\varpi_{i,j}$ is a given family of scalars.

As an associative algebra $DB_H(\mathfrak{e}', \mathfrak{f}')$ is unital generated by the elements E_i, F_i, $(P_\mu)_{\mu\in\mathbb{Z}[I]}$ and H, subject to the following relations:

$$P_\mu P_\nu = \delta_{\mu,\nu} P_\mu,\ hE_i = \chi_i(h_1)E_i h_2,\ F_i h = \overline{\chi}_i(h_2)h_1 F_i,$$
$$P_\mu h = hP_\mu,\ P_\mu E_i = \delta_{i,\mu} E_i P_\mu,\ F_j P_\mu = \delta_{i,\mu} P_\mu F_j,$$

$$[F_j, E_i] =$$
$$= \big(\sum_{\mu,\nu} q^{-\mu\cdot\nu}\chi_i'(P_\mu \otimes \overline{R}^2)P_\nu \overline{R}^1 - \sum_{\mu,\nu} q^{\mu\cdot\nu}\overline{\chi}_i'(P_\nu \otimes R^1)P_\mu R^2\big)\delta_{\chi_i,\overline{\chi}_j}\varpi_{i,j}$$
$$= \big(\sum_{\mu} q^{-i\cdot\mu}\chi_i(\overline{R}^2)P_\mu \overline{R}^1 - \sum_{\nu} q^{i\cdot\nu}\chi_i(R^1)P_\nu R^2\big)\delta_{\chi_i,\overline{\chi}_j}\varpi_{i,j}$$
$$= \big(\sum_{\mu} q^{-i\cdot\mu}P_\mu \mathfrak{G}_i - \sum_{\nu} q^{i\cdot\nu}P_\nu \mathfrak{K}_i\big)\delta_{\chi_i,\overline{\chi}_j}\varpi_{i,j},$$

for all $i, j \in I$

and the Serre's relations: i) $E_i E_j = E_j E_i$, for all $i \neq j$ such that

$$\sum_{\mu,\nu} \chi_i'(q^{\mu\cdot\nu} P_\mu \otimes R^1)\chi_j'(P_\nu \otimes R^2) =$$

$$\sum_{\mu,\nu} \chi_j'(q^{\mu\cdot\nu} P_\mu \otimes R^1)\chi_i'(P_\nu \otimes R^2) = 1$$

$$\Leftrightarrow\; q^{i\cdot j}\chi_i(R^1)\chi_j(R^2) = q^{i\cdot j}\chi_i(R^1)\chi_j(R^2) = 1$$

$$\Leftrightarrow\; \chi_i(R^1)\chi_j(R^2) = \chi_j(R^1)\chi_i(R^2) = q^{-i\cdot j}.$$

ii) $E_iE_iE_j - [2]E_iE_jE_i + \chi_i(x^1x^3)\chi_j(x^2)E_jE_i^2 = 0$, for all $i \neq j$ such that

$$\sum_{\mu,\nu\in\mathbb{Z}[I]} q^{\mu\cdot\nu}\chi_i'((P_\mu \otimes R^1)(P_\nu \otimes R^2)) = q^2 \;\Leftrightarrow\; q^{i\cdot i}\chi_i(R^1R^2) = q^2$$

$$\Leftrightarrow\; \chi_i(R^1R^2) = q^{2-i\cdot i}$$

and

$$\sum_{\mu,\nu\in\mathbb{Z}[I]} q^{\mu\cdot\nu}\chi_i'(P_\mu \otimes R^1)\chi_j'(P_\nu \otimes R^2) =$$

$$\sum_{\mu,\nu\in\mathbb{Z}[I]} q^{\mu\cdot\nu}\chi_j'(P_\mu \otimes R^1)\chi_i'(P_\nu \otimes R^2) = q^{-1}$$

$$\Leftrightarrow \chi_i(R^1)\chi_j(R^2) = \chi_j(R^1)\chi_i(R^2) = q^{-1-i\cdot j}.$$

iii) the relations for the F_i's, analogous with those in ii) for the E_i's.

A concrete example: the cyclic group

Let $C_n = \langle K \rangle$ be the cyclic group of order n written multiplicatively, and k a field that contains a primitive n^2 root of unity, say ζ, such that $\zeta^{2n} = 1$. For $\gamma = \zeta^n$,

$$\Phi = \sum_{i,j,l=0}^{n-1} \gamma^{i[\frac{j+l}{n}]} 1_i \otimes 1_j \otimes 1_l.$$

is a normalized 3-cocycle that endows $k[C_n]$ with a quasi-Hopf algebra structure (denoted by $k_\Phi[C_n]$); here, for any $0 \leq j \leq n-1$,

$$1_j = \frac{1}{n} \sum_{i=0}^{n-1} \gamma^{(n-j)i} K^i$$

.

Furthermore, $(k_\Phi[C_n], R)$ is a quasitriangular quasi-Hopf algebra with $R = \sum_{u,v} \zeta^{uv} 1_u \otimes 1_v$.

Take $\chi_i(K) = \gamma^{m_i}$ and $\overline{\chi}_i(K) = \gamma^{n_i}$, where $m_i, n_i \in \mathbb{N}_{<n}$. Then $\mathfrak{K}_i = K^{m_i}$ and $\mathfrak{G}_i = K^{n_i}$ and for this datum one can consider the quasi-quantum group $(\mathfrak{e}' \times (k^{\mathbb{Z}[I]} \otimes k_\Phi[C_n]) \times \mathfrak{f}')^{\widehat{\sigma}}$.

Note that the Serre relations read in this case as

$$\text{since } \chi_i(R^1)\chi_j(R^2) = \chi_j(R^1)\chi_i(R^2) = \gamma^{m_i n_j} \Rightarrow \ \gamma^{m_i n_j} = q^{-i\cdot j};$$
$$\text{since } \chi_i(R^1 R^2) = \gamma^{m_i^2} \Rightarrow \ \gamma^{m_i^2} = q^{2-i\cdot i} \text{ and } \gamma^{m_i n_j} = q^{-1-i\cdot j}.$$

These can be reduced to $i \cdot j = 0$, and respectively to $i \cdot i = 2$ and $i \cdot j = -1$, provided that $n \mid m_i n_j$ and $n \mid m_i^2$ (we can always do this, by taking appropriate m_i's and n_j's!).

Symplectic fermion quasi-Hopf algebra

Let $\mathbb{C}$ be the field of complex numbers, N a non-zero odd natural number and $q \in k$ such that $q^2 = -i$. The family of symplectic fermion quasi-Hopf algebras, denoted in what follows by $\mathcal{O}_q(N)$, were introduced in [4]. To have the simplest description for the Yetter-Drinfeld coalgebras derived from $\mathcal{O}_q(N)$, in what follows we will work with a slightly deformed version of $\mathcal{O}_q(N)$ (relative to the presentation of the $\mathcal{O}_q(N)$ given in [5]).

As an algebra, $\mathcal{O}_q(N)$ is the $\mathbb{C}$-algebra generated by K and the families $\{f_j^{\pm} \mid 1 \leq j \leq N\}$, with relations, $1 \leq j, t \leq N$,

$$f_j^{\pm} K = -K f_j^{\pm},\ f_j^{+} f_t^{-} + f_t^{-} f_j^{+} = \delta_{j,t} e_1,\ f_j^{\pm} f_t^{\pm} = -f_t^{\pm} f_j^{\pm},\ K^4 = 1; \tag{22}$$

here $e_1 := \frac{1}{2}(1 - K^2)$.

[4] V. Farsad, A. M. Gainutdinov, I. Runkel, Adv. Math. **400** (2022), 108–247.
[5] J. Berger, A. M. Gainutdinov, I. Runkel, J. Alg. **548** (2020), 96–119.

The comultiplication Δ and counit ε of $\mathcal{O}_q(N)$ are determined by

$$\Delta(K) = K \otimes K,\ \varepsilon(K) = 1, \tag{23}$$

$$\Delta(f_j^{\pm}) = f_j^{\pm} \otimes 1 + \omega_{\pm} \otimes f_j^{\pm},\ \varepsilon(f_j^{\pm}) = 0\ ,\ \forall\ 1 \leq j \leq N, \tag{24}$$

extended to the whole $\mathcal{O}_q(N)$ as unital algebra morphisms; $\omega_{\pm} := (e_0 \pm ie_1)K,\ e_0 := \frac{1}{2}(1 + K^2)$.

The reassociator Φ of $\mathcal{O}_q(N)$ and its inverse Φ^{-1} are given by

$$\Phi = 1 \otimes 1 \otimes 1 + e_1 \otimes e_1 \otimes (K - 1),\ \Phi^{-1} = 1 \otimes 1 \otimes 1 + e_1 \otimes e_1 \otimes (K^3 - 1) \tag{25}$$

where $\beta_{\pm} := e_0 + q^2(\pm i)^N e_1 K^N$.

Note that our reassociator for $\mathcal{O}_q(N)$ differs from their reassociator.

Another different structure that we consider for $\mathcal{O}_q(N)$ is the triple that defines the antipode of the quasi-Hopf algebra $\mathcal{O}_q(N)$. In our definition, the antipode of $\mathcal{O}_q(N)$ is determined by

$$S(K) = K^{(-1)} = (e_0 - e_1)K,\ S(f_j^{\pm}) = f_j^{\pm}(e_0 \pm i e_1)K,\ \alpha = \beta_+,\ \beta = 1, \tag{26}$$

with S extended to the whole $\mathcal{O}_q(N)$ as an anti-morphism of algebras.

Consider the classical reassociator for $k[C_4]$ given by

$$\Phi_2 = \sum_{a,b,c=0}^{3} (-1)^{a[\frac{b+c}{4}]} 1_a \otimes 1_b \otimes 1_c.$$

There is a twist F such that $(\Phi_2)_F = \Phi$. As for $k_{\Phi_2}[C_4]$ we have a natural QT-structure given by $R_2 = \sum_{u,v} i^{uv} 1_u \otimes 1_v$, we get for free an R-matrix R, $R = (R_2)_F$, for $k\langle K \rangle$, provided that $q^2 = -i$. In the basis given by the powers of K,

$$R = \frac{1}{8}\begin{pmatrix} 3+q & 2-(1+q)i & 1-q & 2+(1+q)i \\ 2+(1+q)i & -(1+q) & (1-q)i & -1-2i+q \\ 1-q & -(1-q)i & -(1-q) & (1-2)i \\ 2-(1-q)i & -1+2i+q & -(1-q)i & -(1+q) \end{pmatrix}.$$

We don't have $\mathcal{O}_q(N) = (\mathfrak{e}' \times k\langle K\rangle \times \mathfrak{f}')^{\widehat{\sigma}}$, but one can show that $\mathcal{O}_q(N) = ((\mathfrak{e}\widetilde{\otimes}\bar{\mathfrak{f}}) \times k\langle K\rangle)_\Sigma$. Actually, we computed the quasi-Hopf algebra structure of $(\mathfrak{e} \times k\langle K\rangle \times \mathfrak{f})^{\widehat{\sigma}}$ and transported to $((\mathfrak{e}\widetilde{\otimes}\bar{\mathfrak{f}}) \times k\langle K\rangle)_\Sigma$ through the natural isomorphism $((\mathfrak{e}\widetilde{\otimes}\bar{\mathfrak{f}}) \times k\langle K\rangle)_\Sigma \simeq (\mathfrak{e} \times k\langle K\rangle \times \mathfrak{f})^{\widehat{\sigma}}$, and so we landed at the infinite dimensional version of $\mathcal{O}_q(N)$: we don't get the relations $f_j^\pm f_t^\pm = -f_t^\pm f_j^\pm$; one can add them by factorizing with the ideal generated by them, a quasi-Hopf ideal actually.

We can also consider the Serre relations for the infinite dimensional version of $\mathcal{O}_q(N)$, by following the same idea as in the cyclic case presented above (by taking $m_i = n_j = 2$, for all $i, j \in I$). Of course, $\mathfrak{e}$ is the set of f_i^-'s and $\mathfrak{f}$ is the set of f_i^+'s

THANK YOU!!!