Ideas to construct infinite-genus minimal surfaces

Hao Chen
ShanghaiTech University, China

Infinite-genus minimal surfaces are important due to applications in material sciences. But there have been very few successful constructions for such surfaces. I will review the few existing examples and propose some ideas for future constructions, some very promising and some speculative. I will point out the specific technical difficulties and hope to stimulate discussions and new insights from the audience.

Reflections on Bubbles and Films in Spaceforms

Rob Kusner
University of Massachusetts at Amherst, USA

Reflection symmetries have been key to understanding the geometry of complete minimal and constant mean curvature (CMC) surfaces embedded in space forms since the work of Alexandrov in the 1950s. Three decades later, these methods --- combined with homological balancing or conservation laws inspired by Noether --- helped reveal the asymptotic structure of such surfaces and led to the development of their moduli space theory. In particular, the classification of all coplanar CMC surfaces in $\mathbf{R}^3$ --- those with a reflection symmetry whose fundamental domain is a graph over a planar domain --- proceeded by understanding the sub-moduli space of all coplanar surfaces in terms of complete $\mathbf{C}P^1$-structures on $\mathbf{C}$, using subtle ideas from Teichmüller theory. We will quickly review our longstanding investigation of these moduli spaces, then turn to our recent (joint, with Karpukhin, McGrath, and Stern) work constructing free boundary minimal surfaces of every topology embedded in $\mathbf{B}^3$, along with many more closed embedded minimal surfaces of area $< 8\pi$ in $\mathbf{S}^3$, and (most recently) the construction of nonorientable embedded minimal surfaces in $\mathbf{S}^4$ of every topological type with area $< 8\pi$. Note that, aside from the Veronese $\mathbf{R}P^2$, none of these surfaces (nor their orientable double covers) can be ``superminimal" in $\mathbf{S}^4$.

A characterization of the Lawson minimal surface of genus two

Joaquín Pérez
University of Granada & Institute of Mathematics IMAG, Spain

The isometry group of the classical Lawson embedded minimal surface $\xi_{2,1} \subset \mathbb{S}^3$ of genus 2 is isomorphic to the group O_{48} of isometries of a regular octahedron, of order 48. O_{48} has a subgroup of index 3 isomorphic to the bidihedral group $D_{4h} = \mathbb{Z}_2 \times D_4$, where D_4 is the dihedral group of order 8 . We will explain how to prove that $\xi_{2,1}$ is the unique closed embedded minimal surface of genus 2 in $\mathbb{S}^3$ whose isometry group contains D_{4h}. This is a joint work with José Espinar.

CMC hypersurfaces of finite index and progress on Do Carmo's problem

Han Hong
Beijing Jiaotong University, China

Do Carmo raised a question in his lecture notes whether a complete noncompact stable constant mean curvature hypersurface in $\mathbb{R}^n$ (or even with finite index) must be minimal. Affirmative answer in $\mathbb{R}^n$ for $n = 3, 4, 5$ and partial results in higher dimensions have been proved in the last three decades. In this talk, we will show that it is also true in $\mathbb{R}^6$. The proof is also applicable to lower dimensions, thereby providing alternative proofs for those previously resolved cases. We will also talk about some progress in hyperbolic space if time permits. This talk is based on a joint work with Jingche Chen and Haizhong Li.

On the Loop Group Construction of Minimal Surfaces in the Heisenberg Group and Its Applications

Shimpei Kobayashi
Hokkaido University, Japan

The study of minimal surfaces in the Heisenberg group has been a central topic in the context of surface theory in Thurston geometries. In this talk, we discuss a construction of such surfaces using loop group techniques, an approach based on integrable systems.

Analogous to the classical Weierstrass representation for constant mean curvature surfaces in Euclidean space, the minimal surface equation in the Heisenberg group can be described by harmonic maps into a symmetric space. We will explain how the loop group decomposition allows us to construct these surfaces systematically from holomorphic data.

Furthermore, we will present recent applications of this construction, including the generation of new examples and the analysis of their global geometric properties.

Orthogonal ring patterns and discrete constant mean curvature surfaces

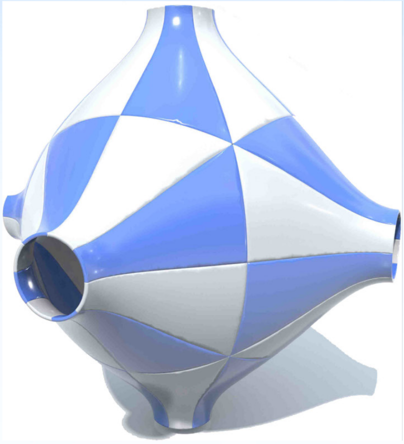
Alexander I. Bobenko
Technische Universität Berlin, Germany

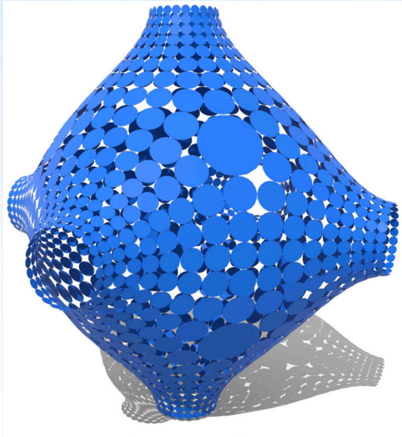
We introduce orthogonal ring patterns consisting of pairs of concentric circles. They generalize orthogonal circle patterns which can be treated as a conformal limit. It is shown that orthogonal ring patterns in Euclidean and hyperbolic planes and in a sphere are governed by integrable equations. We deliver variational principles which are used to prove existence and uniqueness results, and also to compute ring patterns with classical boundary conditions. The later are used to generate discrete constant mean curvature surfaces. See the figure demonstrating a pair of corresponding discrete and smooth CMC surfaces. The relation to minimal surfaces in S^3 and AdS_3 is discussed. Numerous models will be presented.

The talk is based on the recent papers:

A.I. Bobenko, Spherical and hyperbolic orthogonal ring patterns: Integrable systems and variational principles, Trans. AMS (2025)

A.I. Bobenko, T. Hoffmann, N. Smeenk, Discrete constant mean curvature surfaces and orthogonal ring patterns. Geometry from combinatorics, arXiv:2410.08915 (2024)





Geometry and Dynamics of mean curvature flows

Jinxin Xue
Department of Mathematics, Tsinghua University, China

Mean curvature flow is to evolve an embedded hypersurface in Euclidean space according to the vector field given by the mean curvature at each point. We show how to extract geometric information from the asymptotic dynamics of a mean curvature flow approaching a singularity. We show that when the flow approaches a cylinder in a nondegenerate manner, i.e. with the slowest asymptotic rate, the flow can be extended through the singularity and the topology change can be described explicitly. When the flow approaches a cylinder in a degenerate manner, i.e. with exponential asymptotic rate, the singularity set can be proved to be at least C^2 modulo a lower dimensional set. This talk is based on a series of works jointly with Ao Sun and Zhihan Wang.

Willmore surfaces: geometry, analysis and integrable system

Peng Wang
Fujian Normal University, China

In this talk, we will introduce our recent works on Willmore surfaces via various methods.

For Willmore 2-spheres in S^n, we give a structural description. For minimal tori in S^4, we obtain a Morse index estimate. For genus-g embedded surfaces in S^3 with given symmetries, we show that the Lawson minimal surface $\xi_{g,1}$ minimizes the Willmore energy uniquely. By integrable system methods, we obtain a Willmore deformation of Willmore surfaces in S^n. In particular, a complete minimal Möbius strip in $\mathbb{H}^4$ is constructed by deforming the Veronese surface in S^4. The talk is based on various joint works separately with Professors D. Brander, J. Dorfmeister, R. Kusner, Y. Lv, X. Ma, F. Pedit and C.P. Wang.

Understanding equivariant minimal surfaces using Higgs bundles

Ian McIntosh
University of York, United Kingdom

Over a series of papers (some of which were joint work with J Loftin, Rutgers-Newark) we have developed an approach to understanding the moduli space of equivariant minimal surfaces in real and complex hyperbolic spaces which uses G Higgs bundles where G is the isometry group of the space. This approach is especially fruitful for low dimensions (real hyperbolic 3 and 4 space, the complex hyperbolic plane). The aim of this talk is to explain this approach and some of the consequences, as well as describe some open problems.

On the variation of the Willmore functional for surfaces in 4-dimensional conformal manifolds

Zhenxiao Xie
Beihang University, China

In this talk, the Willmore functional is defined as the squared L^2-norm of the trace-free part of the second fundamental form. It is a global conformal invariant. We first derive its Euler-Lagrange equation in a conformally invariant form. Examples of Willmore surfaces in certain Kähler-Einstein surfaces are discussed. We then present two applications of the second variation. First, we prove that the Clifford torus in $\mathbb{C}P^2$ is strictly Willmore-stable. This provides a strong support to the conjecture of Montiel and Urbano, which asserts that the Clifford torus minimizes the Willmore functional among all tori or all Lagrangian tori in $\mathbb{C}P^2$. Second, for complex curves in Kähler surfaces, we establish a lower bound on the first eigenvalue of the area Jacobi operator. This talk is mainly based on a joint work with Prof. Changping Wang.

Minimal Isotropic Surfaces in the Complex Projective Space

Andrey Mironov
Sobolev Institute of Mathematics, Novosibirsk, Russia

Using Baker-Akhiezer function, we construct minimal isotropic surfaces in complex projective space. We show that the Novikov-Veselov equations define deformations of such surfaces preserving the spectral curve of the Schrödinger operator associated with the surface. We also show that the Novikov-Veselov equations define symmetries of some soliton equations related to the minimal surfaces. The results were obtained with Meyramgul Ermentay and Hui Ma.

Index and total curvature of minimal surfaces in noncompact symmetric spaces and wild harmonic bundles

Qiongling Li
Nankai University, Chern Institute of Mathematics, China

We prove two main theorems about equivariant minimal surfaces in arbitrary nonpositively curved symmetric spaces extending classical results on minimal surfaces in the Euclidean space. First, we show that a complete equivariant branched immersed minimal surface in a nonpositively curved symmetric space of finite total curvature must be of finite Morse index. It is a generalization of the theorem by Fischer-Colbrie, Gulliver-Lawson, and Nayatani for complete minimal surfaces in Euclidean space. Secondly, we show that a complete equivariant minimal surface in a nonpositively curved symmetric space is of finite total curvature if and only if it arises from a wild harmonic bundle over a compact Riemann surface with finite punctures. Moreover, we deduce the Jorge-Meeks type formula of the total curvature and show that it is an integer multiple of $2\pi/N$ for N only depending on the symmetric space. It is a generalization of the theorem by Chern-Osserman for complete minimal surfaces in the Euclidean n -space. This is joint work with Takuro Mochizuki (RIMS).

Constructing isometric tori with the same curvatures

Andrew Sageman-Furnas
Department of Mathematics, North Carolina State University, USA

A longstanding problem in differential geometry, known as the Bonnet problem, asks: is a compact surface in Euclidean three-space uniquely determined by its metric and mean curvature function? The answer is known to be yes for a topological sphere and yes for a generic surface.

In this talk, we explicitly construct a pair of immersed tori that are related by a mean curvature preserving isometry. These tori are the first examples of compact Bonnet pairs. Moreover, we prove these isometric tori are real analytic. This resolves a second longstanding open problem on whether real analyticity of the metric already determines a unique compact immersion.

We describe the discovery of these analytic tori using discrete differential geometry. It involves exploring immersions of a 5×7 quad decomposition of a torus and a theory of discrete Bonnet pairs.

The smooth/analytic theory is joint work with Alexander Bobenko and Tim Hoffmann, and the discrete theory is joint work with Tim Hoffmann and Max Wardetzky.

The existence of constrained Willmore surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$

Ross Ogilvie
Universität Mannheim, Germany

The Willmore energy of an immersion of a closed surface is the integral of the square of its mean curvature. This is a measure of how far a surface is from being a sphere and is a conformal invariant of the immersion. A constrained Willmore surface is a critical point of this energy functional under deformations that preserve the conformal class of the surface. By describing a surface in terms of "holomorphic" data, which are generalizations of the Kodaira and Weierstrass representations, and formulating a corresponding weak problem, it becomes possible to take limits of sequences of immersions and prove the existence of minimizers in each conformal class.

On the enclosed value of constant mean curvature surfaces in 3-space

Lynn Heller
BIMSA, China

We derive a general formula for the enclosed volume of CMC surfaces in $\mathbb{R}^3$ with translational periods in a lattice in terms of area and the Wess-Zumino-Witten term generalising the well-known Minkowski formula for compact CMC surfaces. We then show how to apply the formula to obtain values for the enclosed volume in various examples by expressing it as a gauge invariant quantity. This allows us to give a numerical counter example for the periodic isoperimetric problem, where the minimizers were conjectured to be spheres, cylinders or two planes. This is a joint work with Sebastian Heller and Martin Traizet.

Minimal surfaces with parallel distributions of planes via cyclic Higgs bundles

Samuel Bronstein
MPI-MIS Leipzig, Germany

We exhibit via Higgs bundles equivariant minimal surfaces in symmetric spaces of high rank with special geometry, that is with parallel distributions over it. Using this parallelity, we are able under certain circumstances to show that these minimal surfaces are quasi-isometrically embedded, and that the associated surface group representations have some Anosov property, depending on the parallel family of objects. This work is a collaboration with Prof. Qiongling Li (Chern Institute) and Dr. Colin Davalo (Torino University).

Weierstrass representation of Proper Definite Affine Spheres (or PDAS-surfaces)

Erxiao Wang
Zhejiang Normal University, China

PDAS-surfaces in $\mathbb{R}^3$ are equivalent to primitive harmonic maps from Riemann surfaces into the 6-symmetric space $\mathrm{SL}_3\mathbb{R}/\mathrm{SO}_2\mathbb{R}$. After proving an Iwasawa-type double coset decomposition for the associated complex twisted loop group of $A_2^{(2)}$ type, we give a Weierstrass-type representation by either holomorphic or other meromorphic DPW potentials. The linear DPW flow on the only two open Iwasawa cosets corresponds exactly to the elliptic and the hyperbolic parts respectively, while the other cosets having finite codimensions provide algebraic types for the branch curves or points between these parts. Then equivariant PDAS-surfaces are classified: among all seven Riemann surfaces admitting one-parameter group of automorphisms, the Riemann sphere, the complex plane, the punctured plane, the unit disk and the annulus admit affine conformal branch immersions into $\mathbb{R}^3$ as global equivariant PDAS-surfaces; but the punctured disk and any torus do not admit such immersions. There is no equivariant PDAS-surface with screw-motion symmetry either. Most of these global surfaces consist of both elliptic and hyperbolic parts. They suggest that the Calabi conjecture for complete hyperbolic affine spheres (proved by Cheng-Yau) may be generalized to some global PDAS-surfaces of mixed parts. This is a joint work with Josef Dorfmeister and Gang Wang.

Totally complex submanifolds and R-spaces associated with quaternionic Kähler symmetric spaces

Yoshihiro Ohnita
Waseda University & OCAMI Osaka Metropolitan University, Japan

This talk is based on my joint work with Jong Taek Cho (Chonnam Natl. U., Korea) and Kaname Hashimoto (OCAMI/Bunkyo Univ., Japan).

Totally complex submanifolds of quaternionic Kähler manifolds form a distinguished class of minimal submanifolds. They are minimal submanifolds covered by horizontal and complex submanifolds in the associated twistor spaces. In this work we construct a new (non Levi-Civita) canonical connection on the inverse image of any maximal dimensional totally complex submanifold of the quaternionic projective space $\mathbb{H}P^n$ under the Hopf fibration $S^{4n+3}(1) \to \mathbb{H}P^n$. As its application, we provide a geometric proof based on the theorem of Olmos and Sánchez (J. reine angew. Math. 1991) that any maximal dimensional totally complex submanifold of $\mathbb{H}P^n$ with parallel second fundamental form arises as the projection of a certain singular R-space associated with a quaternionic Kähler symmetric space under the Hopf fibration. Utilizing the Lie algebraic structure of quaternionic Kähler symmetric pairs, we determine such R spaces explicitly and, in particular, recover all maximal dimensional totally complex submanifolds of $\mathbb{H}P^n$ with parallel second fundamental form, previously classified by Tsukada (Osaka J. Math. 1985) using a different method. Moreover, we investigate additional geometric properties of these totally complex submanifolds, such as their fundamental groups, weak reflectivity, associated moment maps and Lagrangian submanifolds.

Higgs Bundles and SYZ Geometry

Charles Ouyang
Washington University in St. Louis, USA

Special Lagrangian 3 -torus fibrations over a 3 -dimensional base play an important role in mirror symmetry and the SYZ conjecture. In this talk, we discuss the construction, via Higgs bundles, of an infinite family of semi-flat Calabi-Yau metrics on special Lagrangian torus bundles over an open ball in $\mathbb{R}^3$ with a Y -vertex deleted. This is a joint work with S. Heller and F. Pedit.

Volume preserving curvature flows for hypersurfaces

Yong Wei
University of Science and Technology of China, China

I will describe some recent work (partly with Ben Andrews, Xuzhong Chen, Bo Yang, and Tailong Zhou) on volume-preserving geometric flows for hypersurfaces and their applications to proving geometric inequalities. These flows are governed by the kth mean curvature of the evolving hypersurface and can be interpreted as gradient flows for quermassintegrals. Under convexity assumptions, we analyze the convergence of such flows in both Euclidean and hyperbolic spaces. Key techniques employed in our proofs include the (tensor/vector bundle) maximum principle and the curvature measure theory from convex geometry.

Rigidity on capillary CMC and minimal hypersurfaces

Chao Xia
Xiamen University, China

In this talk, I give a survey on recent rigidity results on capillary CMC and minimal hypersurfaces in Euclidean balls and half-spaces, including classification of stable capillary CMC, embedded capillary CMC and capillary minimal graphs with linear growth. We will also mention related results in anisotropic setting and in Gaussian space setting.

Minimal surfaces, mKdV hierarchy and the Serrin problem

Pablo Mira
Universidad Politecnica de Cartagena, Spain

In 1971, J. Serrin showed that the only smooth bounded domains $\Omega \subset \mathbb{R}^n$ where the overdetermined problem

$$\begin{cases} \Delta u + 2 = 0 & \text{in} \quad \Omega \\ u = 0, \quad \frac{\partial u}{\partial \nu} = b & \text{on} \quad \partial\Omega \end{cases}$$

can be solved are balls. The proof of this result was very influential in elliptic PDE theory, as it introduced the geometric Alexandrov reflection principle to this context.

In this talk we will introduce integrable systems theory to the study of Serrin's problem, based on ideas from minimal and CMC surfaces. We will show that any annular domain $\Omega \subset \mathbb{R}^2$ where $\Delta u + 2 = 0$ can be solved with locally constant overdetermined boundary conditions is of finite type for the mKdV hierarchy. The same result holds for periodic bands. By analyzing low spectral genus, we will recover all previously known examples, obtained by analytic bifurcation or desingularization techniques, and place them inside a global moduli space of explicit solutions. The Serrin planar domains we construct are natural overdetermined analogues of wellknown minimal and CMC surfaces, like Wente tori or Riemann's minimal examples. Joint work with Alberto Cerezo and Isabel Fernandez.

Lu's conjecture for minimal surfaces

Jianquan Ge
Beijing Normal University, China

Abstract: After Chern's conjecture on the discreteness of the constant scalar curvatures of compact minimal submanifolds M^n in unit spheres $\mathbb{S}^{n+q}$, Z. Q. Lu proposed a conjecture about the second gap which is based on his ingenious improvement of the known first gap theorem: If $0 \leq S + \lambda_2 \leq n$, then $S + \lambda \equiv 0$ or n, and thus $S \equiv 0, n,$ or $\frac{2}{3}n$, which unifies Simons' first gap of hypersurfaces and Yau-Shen-Li-etc's first gap of high codimensional submanifolds. Here S is the squared norm of the second fundamental form and λ_2 is the second largest eigenvalue of the fundamental matrix $A := (\langle S_\alpha, S_\beta \rangle)_{q \times q}$ for the shape operators $\{S_\alpha\}_{\alpha=1}^q$.

Lu's conjecture states that: If $S + \lambda_2 \equiv$ Const $> n$ then there is a constant $\varepsilon(n, q) > 0$ such that $S + \lambda_2 > n + \varepsilon(n, q)$.

For hypersurfaces, second gaps have been shown exist though not optimal both under constant and nonconstant scalar curvature assumption by many authors. However, for high codimensional submanifolds, both of Chern's conjecture and Lu's conjecture have no progress on the second gap. For 2-dimensional minimal submanifolds, Chern's conjecture has been verified by Calabi for 2 -spheres and by Bryant for general surfaces; however, as a generalization, Lu's conjecture has not any progress.

In this work joint with Weiran Ding, Fagui Li and Xize Yang, for arbitrary codimension, we prove Lu's conjecture for minimal 2 -spheres, and for any minimal surfaces under some slight inequality condition about the normal scalar curvature.