Generalized Symmetries from Fusion Actions

Li Ren

Sichuan University

Joint work with Chongying Dong, Siu-Hung Ng and Feng Xu

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Content

1. Introduction

Orbifold theory

- A is a simple vertex operator algebra
- G is a finite automorphism group of A
- Orbifold theory: Study the A^G-module category

Results in orbifold theory relevant to this talk

1. Schur-Weyl duality
2. Galois correspondence
3. A complete Galois correspondence

1. Introduction

$\mathrm{irr}(G)$: irreducible characters of G

W_λ: irreducible G-module affording to $\lambda \in \mathrm{irr}(G)$

Schur-Weyl duality[Dong-Li-Mason 96, Kac-Todorov 97]

1. $A = \oplus_{\lambda \in \mathrm{irr}(G)} W_\lambda \otimes A_\lambda$ where A_λ is the multiplicity space of W_λ in A
2. $\{A_\lambda \mid \lambda \in \mathrm{irr}(G)\}$ are inequivalent irreducible A^G-modules

Remark

The duality result holds for any compact Lie group acting continuously on A

1. Introduction

Galois crespondence[Dong-Mason 97, Hanaki-Miyamoto-Tambara 99]

Let A be a simple VOA and G a finite automorphism group of A. Then

$$H \mapsto A^H$$

gives a one to one correspondence from the subgroups of G to the sub VOAs of A containing A^G

Galois correspondence[Dong-Jiao-Xu 2013]

If we assume further that A is rational, C_2-cofinite and the weight of any irreducible twisted module is positive except A itself, then $\dim_{A^G} A = \frac{\dim A}{\dim A^G} = o(G)$ (no category theory involved)

1. Introduction

Problem: Not every sub VOA B of A can be realized as A^G for some group G

Questions

- Is there a replacement for G such that B arises as a fixed point of some action?
- Does VOA A have generalized symmetries beyond group action?

Answer

- Yes, there is a fusion category $\mathcal{F}$ acting on A such that $B = A^{\mathcal{F}}$. The fusion action gives generalized symmetries
- In fact, there is fusion action on any condensable algebra A in a modular tensor category $\mathcal{C}$!

2. Fusion actions

Notations

- $\mathcal{F}$: Fusion category
- $\mathrm{Irr}(\mathcal{F})$: equivalence classes of simple objects
- $\mathcal{F}(x, y)$: morphism space from x to y for $x, y \in \mathcal{F}$
- $K(\mathcal{F})$: the fusion algebra over $\mathbb{C}$ which is a semisimple associative algebra
- $\mathcal{C}$: modular tensor category (MTC)

2. Fusion actions

Condensible algebra

$A \in \mathcal{C}$ is called a condensable algebra:

- A is an algebra: $m_A : A \otimes A \to A$, $u_A : \mathbf{1} \to A$
- A is connected: $\dim \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) = 1$
- A is commutative: $m_A = m_A R_{A,A}$ where $R_{A,A} : A \otimes A \to A \otimes A$ is the braidng
- $\dim A \neq 0$
- $\theta_A = 1$
- $\epsilon m_A : A \otimes A \to \mathbf{1}$ is nondegenerate where $\epsilon \in \mathcal{C}(A, \mathbf{1})$ denotes the section of u_A $(A \cong A^*)$

2. Fusion actions

A is a condensable algebra

A-modules

- $M \in \mathcal{C}$ is a right A-module: $m_M : M \otimes A \to M$
- Right A-module M is called local module: $\theta_M = \lambda \mathrm{id}_M$
- A-module category $\mathcal{C}_A$ is a fusion category
- Local A-module category $\mathcal{C}_A^0$ is a MTC
- $\alpha(x) = x \otimes A \in \mathcal{C}_A$ for $x \in \mathcal{C}$
- Any simple object of $\mathcal{C}_A$ is a subobject of $\alpha(x)$ for some simple $x \in \mathcal{C}$

2. Fusion Actions

Fix a MTC $\mathcal{C}$ and a condensable algebra $A \in \mathcal{C}$. Then

$$A = \oplus_{x \in \mathrm{Irr}(\mathcal{C})} W_x \otimes x$$

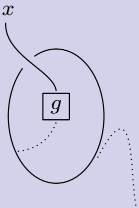
where $W_x = \mathcal{C}(x, A)$

2. Fusion Actions

Fusion action on W_x

For $x \in \mathcal{C}$, $Y \in \mathcal{C}_A$ and $g \in W_x = \mathcal{C}(x, A)$

$$Yg := \frac{1}{d(A)}$$



$$Y \cdot g = \Big(x \xrightarrow{x \otimes \mathrm{coev}_Y} x \otimes Y \otimes Y^* \xrightarrow{R_{x,Y} \otimes Y^*} Y \otimes x \otimes Y^* \xrightarrow{Y \otimes g \otimes Y^*} Y \otimes A \otimes Y^* \xrightarrow{\mu_Y \otimes Y^*} Y \otimes Y^* = Y^{**} \otimes Y^* \xrightarrow{\widetilde{\mathrm{ev}}_{Y^*}} A\Big)$$

2. Fusion Actions

Theorem

- The actions of the fusion category $\mathcal{C}_A$ on W_x satisfies

$$(X \otimes_A Y)f = X(Yf)$$

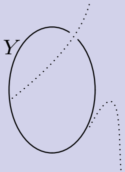
for $X, Y \in \mathcal{C}_A$ and $f \in W_x$. In particular, W_x are modules for $K(\mathcal{C}_A)$

2. Fusion Actions

Fusion action on A

The objects $Y \in \mathcal{C}_A$ acting on A is defined by an algebra homomorphism $K(\mathcal{C}_A) \to \mathcal{C}(A, A)$.

$$\rho(Y) = Y\,\mathrm{id}_A = \frac{1}{d(A)}$$



for $Y \in \mathrm{Irr}(\mathcal{C}_A)$,

where

$$\mathcal{C}(A, A) = \bigoplus_{x \in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(x, A) \otimes \mathcal{C}(A, x)\,.$$

3. Schur-Weyl duality

Definitions and notations

- $e_{\mathbf{1}} = \frac{1}{\dim(\mathcal{C}_A^0)} \sum_{X \in \mathrm{Irr}(\mathcal{C}_A^0)} d_A(X)X$ is a primary idempotent element of $K(\mathcal{C}_A)$
- $e_{\mathbf{1}}K(\mathcal{C}_A)$ is a semisimple ideal of $K(\mathcal{C}_A)$
- Let V be a $K(\mathcal{C}_A)$-module. Define $\mathcal{C}_A$-invariants (fixed points)

$$V^{\mathcal{C}_A} = \{f \in V \mid Xf = d_A(X)f \text{ for } X \in \mathrm{Irr}(\mathcal{C}_A)\}$$

and $A^{\mathcal{C}_A} = \sum_x W_x^{\mathcal{C}_A} \otimes x$

3. Schur-Weyl duality

Theorem[Schur-Weyl duality]

Fix $\mathcal{C}$ and A. Then

1. The kernel of the action is equal to $(1-e_{\mathbf{1}})K(\mathcal{C}_A)$
2. For any $x \in \mathrm{Irr}(\mathcal{C})$ with $W_x \neq 0$, W_x is an irreducible $K(\mathcal{C}_A)$-module
3. $W_x \cong W_y \neq 0$ as $K(\mathcal{C}_A)$-module if and only if $x = y$ in $\mathcal{C}$
4. The restriction $e_{\mathbf{1}}K(\mathcal{C}_A) \to \mathcal{C}(A, A)$ is an algebra isomorphism
5. $A^{\mathcal{C}_A^0} = A$
6. $A^{\mathcal{C}_A} = \mathbf{1}$

3. Schur-Weyl duality

Remark

1. A hypergroup action in the categorical setting was recently introduced by Riesen (2025): if A is an extension of rational VOA B then there is an hypergroup K acting on A such that $A^K = B$. This result is partially related to our result that $A^{\mathcal{C}_A} = \mathbf{1}$
2. Although the duality result for the fusion category action on A is similar to the duality result in orbifold theory, the proof for orbifold theory setting does not work here

3. Schur-Weyl duality

Remark

1. Orbifold theory:

$$A = \oplus_{\lambda \in \mathrm{irr}(G)} W_\lambda \otimes A_\lambda$$

where A_λ is the multiplicity space of the irreducible G-module W_λ in A. Try to understand A_λ as A^G-module (classical invariant theory)

2. Fusion action

$$A = \oplus_{x \in \mathrm{Irr}(\mathcal{C})} W_x \otimes x$$

where W_x is the multiplicity space of x in A. Try to understand W_x as $K(\mathcal{C}_A)$-module (Frobenius-Schur indicators)

3. Schur-Weyl duality

Group action on VOA is a fusion action:

- A : a simple VOA
- G : a finite automorphism group of A such that A^G is rational and C_2-cofinite
- A^G-module category $\mathcal{C} = \mathcal{M}_{A^G}$ is a MTC and $A \in \mathcal{C}$ is a condensable algebra

Theorem

The fusion action of $\mathcal{C}_A$ on A is equivalent to the G-action on A

4.Galois correspondence

Theorem[Galois correspondence]

Let $\mathcal{C}$ be a pseudounitary modular tensor category, and A a condensable algebra in $\mathcal{C}$. Then the assignment $\mathcal{B} \mapsto A^{\mathcal{B}}$, defines a bijection between the fusion subcategories of $\mathcal{C}_A$ containing $\mathcal{C}_A^0$ and subalgebras of A, whose inverse is given by assignment $B \mapsto (\mathcal{C}_B^0)_A$. In particular, $\mathcal{B} = (\mathcal{C}^0_{A^{\mathcal{B}}})_A$ for any $\mathcal{B}$ and

$$\dim(\mathcal{B}) = \frac{\dim(\mathcal{C})}{d(A)d(B)}$$

4. Galois correspondence

Remark

1. Such a correspondence was also established by Davydov–Müger–Nikshych–Ostrik (2013) using the theory of the **relative center**, where $\mathcal{C} = Z(\mathcal{C}_A)$ or A is a Lagrangian algebra in $\mathcal{C}$. In their framework, the subalgebra $B = I(\mathbf{1})$ of A is obtained via the **right adjoint** I of the forgetful functor $F_{\mathcal{B}} : \mathcal{C} \to Z_{\mathcal{B}}(\mathcal{C}_A)$, where $Z_{\mathcal{B}}(\mathcal{C}_A)$ denotes the relative center of $\mathcal{B}$ in $\mathcal{C}_A$. In contrast, our approach follows the **classical fixed-point method** for correspondence results and relies crucially on **Schur–Weyl duality**
2. This result was also given by Xu (2014) in conformal net setting

5. Fusion actions on VOAs

Application to VOA:

- U: simple, rational, C_2-cofinite VOA of CFT type such that the weight of any irreducible U-module is positive except U itself
- $\mathcal{C} = \mathcal{M}_U$: pseudounitary modular tensor category,
- $A \supset U$ (conformal): simple VOA. Then $A \in \mathcal{C}$ is a condensable algebra which has a decomposition

$$A = \oplus_{x \in \mathrm{Irr}(\mathcal{M}_U)} W_x \otimes x$$

5. Fusion actions on VOAs

Theorem

1. For any $x \in \mathrm{Irr}(\mathcal{C})$ with $W_x \neq 0$, W_x is an irreducible $K(\mathcal{C}_A)$-module
2. For any $x, y \in \mathrm{Irr}(\mathcal{C})$, $W_x \cong W_y \neq 0$ as $K(\mathcal{C}_A)$-module if and only if $x = y$
3. The restriction $e_{\mathbf{1}} K(\mathcal{C}_A) \to \mathcal{C}(A, A)$ is an isomorphism of algebras
4. $\mathcal{B} \mapsto A^{\mathcal{B}}$ gives a one to one correspondence between the fusion subcategories of $\mathcal{C}_A$ containing $\mathcal{C}_A^0$ and subVOAs of A containing U. In particular, $A^{\mathcal{C}_A^0} = A$ and $A^{\mathcal{C}_A} = U$

Corollary

If A is a simple VOA and U a rational, C_2-cofinite subVOA of A. Then there are only finitely many sub VOAs between U and A ($\mathcal{C}_A$ has only finitely many fusion subcategories)

5. Fusion actions on VOAs

Fusion actions associated with coset construction

- A: a holomorphic vertex operator algebra
- U, V: rational and C_2-cofinite subVOAs of A with $U^c = V$ and $V^c = U$
- $A = \oplus_{i=0}^{p} U^i \otimes V^i$ as $U \otimes V$-modules, where $\mathrm{Irr}(\mathcal{M}_U) = \{U^i \mid i = 0, ..., p\}$, $\mathrm{Irr}(\mathcal{M}_V) = \{V^i \mid i = 0, ..., p\}$ and $U^0 = U \;\; V^0 = V$
- $\mathcal{M}_U \simeq \overline{\mathcal{M}_V}$ (braided equivalence)[Dong-Ng-Ren 2025]
- Let $\mathcal{C} = \mathcal{M}_{U \otimes V} = \mathcal{M}_U \boxtimes \mathcal{M}_V$. Then

$$\mathrm{Irr}(\mathcal{C}_A) = \{\alpha(U^i \otimes V^0) \mid i = 0, ..., p\}$$

- $K(\mathcal{C}_A)$ is commutative algebra , with irreducible characters $\chi_i(\alpha(U^j \otimes V)) = \frac{s_{i,j}}{\dim U^i}$
- $A^{\mathcal{C}_A} = U \otimes V$

6. Examples

Fusion actions associated to A_n

- L : root lattice type A_n, dual lattice $L^\circ = \cup_{i=0}^n (L + \lambda_i)$
- $\exists$ another lattice K such that $K^\circ = \cup_{i=0}^n (K + \mu_i)$ such that the orthogonal sum $K+L$ is a sublattice of a positive definite even unimodular lattice E
- $E = \cup_{i=0}^n (L + \lambda_i, K + \mu_i)$, $[E : L + K] = n + 1$

Theorem [Dong-Ng-Ren 2025]

The module category $\mathcal{M}_{V_L}$ is braided equivalent to $\overline{\mathcal{M}_{V_K}}$ (reverse category) where V_L is the lattice VOA and $\mathcal{M}_{V_L}$ is the V_L-module category

6. Examples

- τ : an automorphism of V_E acting on $V_{L+\lambda_r} \otimes V_{K+\mu_r}$ by the scalar $e^{\frac{2\pi i r}{n+1}}$
- σ : the involution induced by the -1-isometry of E. For any subspace X of V_E, let $X^{\pm}$ denote its ±1-eigenspaces
- τ, σ generate a dihedral group G of order $2(n+1)$
- $V_L^+ \otimes V_K^+ < V_E^G = (V_L \otimes V_K)^+ < V_L \otimes V_K < V_E$
- $V_L^+ \otimes V_K^+$ is not an orbifold subVOA of V_E
- $\mathcal{C} = \mathcal{M}_{V_L^+} \boxtimes \mathcal{M}_{V_K^+} = \mathcal{M}_{V_L^+ \otimes V_L^+}$ is a MTC
- $A = V_E \in \mathcal{C}$ is a holomorphic VOA and a condensable algebra

6. Examples

$2n$ cases

- $\dim \mathcal{C} = (8n+4)^2$
- $\dim A = \dim \mathcal{C}_A = (8n+4)$
- $|\mathrm{Irr}(\mathcal{C}_A)| = 4n+4$
- $\mathrm{Irr}(\mathcal{C}_A) = D_{2(2n+1)} \cup \{X, Y\}$
- $G = D_{2(2n+1)} =< \tau, \sigma \mid \tau^{2n+1} = \sigma^2 = 1, \sigma\tau\sigma = \tau^{-1} >$
- $H =< \tau >$
- Fusion relation:
  $hX = X, hY = Y$ for $h \in H$
  $gX = Y, gY = X$ for $g \in G \setminus H$
  $X^2 = Y^2 = \sum_{h\in H} h,\ XY = \sum_{g \in G\setminus H} g$

6. Examples

There are two nontrivial fusion subcategories of $\mathcal{C}_A$ which are not groups:

$$\mathcal{F}_X = H \cup \{X\}, \quad \mathcal{F}_Y = H \cup \{Y\}.$$

The corresponding subalgebras of V_E are

$$V_E^{\mathcal{F}_X} = V_L \otimes V_K^+, \quad V_E^{\mathcal{F}_Y} = V_L^+ \otimes V_K.$$

6. Examples

$2n+1$ cases, n is odd

- $\dim \mathcal{C} = (8n+8)^2$
- $\dim A = \dim \mathcal{C}_A = (8n+8)$
- $|\mathrm{Irr}(\mathcal{C}_A)| = 4n+8$
- $\mathrm{Irr}(\mathcal{C}_A) = D_{2(2n+2)} \cup \{X_1, X_2, Y_1, Y_2\}$
- $G = D_{2(2n+2)} =< \tau, \sigma \mid \tau^{2n+2} = \sigma^2 = 1, \sigma\tau\sigma = \tau^{-1} >$
- $H =< \tau >$
- Fusion relations:
  $X_i^2 = Y_i^2 = \sum_{r=0}^{n} \tau^{2r}$,
  $X_1X_2 = Y_1Y_2 = \sum_{r=0}^{n} \tau^{2r+1}$
  $\tau X_1 = X_2, \tau X_2 = X_1, \tau Y_1 = Y_2, \tau Y_2 = Y_1$
  $X_1Y_1 = Y_1X_1 = X_2Y_2 = Y_2X_2 = \sum_{r=0}^{n} \tau^{2r}\sigma$
  $X_1Y_2 = Y_2X_1 = X_2Y_1 = Y_1X_2 = \sum_{r=0}^{n} \tau^{2r+1}\sigma$

6. Examples

There are 10 nontrivial fusion subcategories of $\mathcal{C}_A$ which are not groups:

1. $\mathcal{F}_{X_i} = \{\tau^{2r}, X_i \mid r = 0, ..., n\}, i = 1, 2$
2. $\mathcal{F}_{Y_i} = \{\tau^{2r}, Y_i \mid r = 0, ..., n\}, i = 1, 2$
3. $\mathcal{F}_{X_1,X_2} = H \cup \{X_1, X_2\},\ H =< \tau >$
4. $\mathcal{F}_{Y_1,Y_2} = H \cup \{Y_1, Y_2\}$
5. $\mathcal{F}_{X_1,Y_1} = D^1 \cup \{X_1, Y_1\},\ D^1 =< \tau^2, \sigma >\cong D_{2n+2}$
6. $\mathcal{F}_{X_2,Y_2} = D^1 \cup \{X_2, Y_2\},$
7. $\mathcal{F}_{X_1,Y_2} = D^2 \cup \{X_1, Y_2\},\ D^2 =< \tau^2, \tau\sigma >\cong D_{2n+2}$
8. $\mathcal{F}_{X_2,Y_1} = D^2 \cup \{X_2, Y_1\}$

6. Examples

The corresponding subalgebras of V_E are

- $V_L \otimes V_K^+ + V_{L+\lambda_{n+1}} \otimes V_{K+\mu_{n+1}}^\pm$
- $V_L^+ \otimes V_K + V_{L+\lambda_{n+1}}^\pm \otimes V_{K+\mu_{n+1}}$
- $V_L \otimes V_K^+$
- $V_L^+ \otimes V_K$
- $V_L^+ \otimes V_K^+ + V_{L+\lambda_{n+1}}^\pm \otimes V_{K+\mu_{n+1}}^\pm$

6. Examples

$2n + 1$ cases, n is even

- $\dim \mathcal{C} = (8n + 8)^2$
- $\dim A = \dim \mathcal{C}_A = (8n + 8)$
- $|\mathrm{Irr}(\mathcal{C}_A)| = 4n + 8$
- $\mathrm{Irr}(\mathcal{C}_A) = D_{2(2n+2)} \cup \{X_1, X_2, Y_1, Y_2\}$
- $G = D_{2(2n+2)} =< \tau, \sigma \mid \tau^{2n+2} = \sigma^2 = 1, \sigma\tau\sigma = \tau^{-1} >$
- $H =< \tau >$
- Fusion relations:
  $X_i^2 = Y_i^2 = \sum_{r=0}^{n} \tau^{2r+1}$,
  $X_1 X_2 = Y_1 Y_2 = \sum_{r=0}^{n} \tau^{2r}$
  $\tau X_1 = X_2, \tau X_2 = X_1, \tau Y_1 = Y_2, \tau Y_2 = Y_1$
  $X_1 Y_1 = Y_1 X_1 = X_2 Y_2 = Y_2 X_2 = \sum_{r=0}^{n} \tau^{2r} \sigma$
  $X_1 Y_2 = Y_2 X_1 = X_2 Y_1 = Y_1 X_2 = \sum_{r=0}^{n} \tau^{2r+1} \sigma$

6. Examples

There are 2 nontrivial fusion subcategories of $\mathcal{C}_A$ which are not groups:

$$\mathcal{F}_{X_1,X_2} = H \cup \{X_1, X_2\}, \quad \mathcal{F}_{Y_1,Y_2} = H \cup \{Y_1, Y_2\}$$

The corresponding subalgebras of V_E are

$$V_L \otimes V_K^+, \quad V_L^+ \otimes V_K$$

THANKS