On the classification of finite GK-dimensional Nichols algebras of twisted Yetter-Drinfeld modules over finite abelian groups

Yuping Yang
Southwest University

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outline

Background

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- Andruskiewitsch and Schneider conjectured that R is a Nichols algebra whenever $\dim(H) < \infty$. This has been proved true when G is a finite abelian group, or when the order of G is odd.
- When H is infinite dimensional, R is generally not a Nichols algebra. Instead, it is a post-Nichols algebra, and its dual (when it exists) is a pre-Nichols algebra.

Pre-Nichols algebra and Nichols algebra

Definition

Let V be a braided vector space and let $T(V) = \oplus_{i\geq 0} V^{\otimes i}$ be the tensor algebra (an N-graded braided Hopf algebra by declaring that every nonzero element in V is primitive).

(1) A pre-Nichols algebra $P(V)$ of V is a quotient $T(V)/I$, where I is an N-graded Hopf ideal contained in $\oplus_{i\geq 2} V^{\otimes i}$.

(2) The Nichols algebra $B(V)$ of V is the quotient $T(V)/I(V)$, where $I(V)$ is the maximal N-graded Hopf ideal contained in $\oplus_{i\geq 2} V^{\otimes i}$.

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For each braided vector space V, there are canonical surjective morphisms of braided Hopf algebras:

$$T(V) \twoheadrightarrow P(V) \twoheadrightarrow B(V).$$

Nichols algebras of diagonal type

Definition

A braided vector space V is called of diagonal type if it has a basis $\{X_1, X_2, \ldots, X_n\}$ such that the braiding is determined by

$$\mathcal{R}(X_i \otimes X_j) = q_{ij} X_j \otimes X_i, \ \ q_{ij} \in \Bbbk^*, \forall\, 1 \leq i \leq j \leq n.$$

If V is diagonal type, its Nichols algebra $B(V)$ and pre-Nichols algebras $P(V)$ are also called of diagonal type.

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- Around 2006, Heckenberger developed the theory of Weyl groupoids and arithmetical root systems for Nichols algebras of diagonal type. He then obtained a complete classification of arithmetic root systems.
- Recently, Angiono and García Iglesias proved that a Nichols algebra of diagonal type has finite GK-dimension if and only if its root system is finite, or equivalently, if it is an arithmetic root system.

Twisted Yetter-Drinfeld modules category

Definition

Let G be a finite abelian group and Φ a 3-cocycle on G. A **Yetter-Drinfeld module** V over $(\Bbbk G, \Phi)$ is a G-graded vector space $V = \oplus_{g \in G} V_g$ such that each V_g is a projective G-representation with respect to Φ_g, that is

$$e \triangleright (f \triangleright v) = \Phi_g(e, f)(ef) \triangleright v \quad \forall e, f \in G, \ \ v \in V_g.$$

Here Φ_g is the 2-cocycle on G defined by $\Phi_g(x, y) = \frac{\Phi(g,x,y)\Phi(x,y,g)}{\Phi(x,g,y)}$, $\forall x, y \in G$.

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- ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$: the category of Yetter-Drinfeld modules over $(\Bbbk G, \Phi)$. Each object in it is called a twisted Yetter-Drinfeld module of G.

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- Braiding: Let $V = \oplus_{g\in G} V_g \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$, the braiding $\mathcal{R}$ of V is determined by $\mathcal{R}(X \otimes Y) = e \triangleright Y \otimes X, \;\; X \in V_e, Y \in V_f$.

Classification problem

Problem (The main problem)

Classify all finite GKdimensional pre-Nichols algebras of objects in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$. It is crucial for the classification of pointed coquasi-Hopf algebras of finite GKdimension.

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Minimal nondiagonal object

To study Nichols algebras of nondiagonal type, we start with the simplest case.

Definition

An object $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ is called a **minimal nondiagonal object** if V is nondiagonal and every nonzero proper subobject of V is diagonal.

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A basic fact: If $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ is nondiagonal, then there exists a minimal nondiagonal object $U \subset V$.

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Proposition

A minimal nondiagonal object in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ *is rank* 3*, that is, it is a direct sum of* 3 *simple objects.*

Structure of Minimal nondiagonal object

Proposition

Let $V = V_{g_1} \oplus V_{g_2} \oplus V_{g_3} \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ *be a minimal nondiagonal object such that* $G = G_V = \langle g_1, g_2, g_3 \rangle$. *Then we have*

(1) $\dim(V_{g_1}) = \dim(V_{g_2}) = \dim(V_{g_3}) = n$, *where* $n = |\frac{\Phi_{g_1}(g_2,g_3)}{\Phi_{g_1}(g_3,g_2)}|$.

(2) *for each* $i \in \{1,2,3\}$, V_{g_i} *has a basis* $\{X_1, X_2, \cdots, X_n\}$ *such that*

$$g_i \triangleright X_l = \alpha_i X_l, \quad 1 \leq l \leq n;$$

$$g_j \triangleright X_l = \beta_i (\frac{\Phi_{g_i}(g_j, g_k)}{\Phi_{g_i}(g_k, g_j)})^{l-1} X_l, \quad 1 \leq l \leq n;$$

$$g_k \triangleright X_l = X_{l+1},\ g_k \triangleright X_n = \gamma_i X_1, \quad 1 \leq l \leq n-1.$$

Here $j \neq k \in \{1,2,3\} \setminus \{i\}$, *and* $\alpha_i, \beta_i, \gamma_i \in \Bbbk^*$ *satisfy certain conditions.*

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Since each minimal nondiagonal object is a direct sum of 3 simple objects, we first consider Nichols algebras of simple objects.

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Theorem

Let $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ be a simple twisted Yetter-Drinfeld module with $\dim(V) \geq 2$, and the G-degree of V is g. Then $B(V)$ is finite GK-dimensional if and only if V is one of the following types:

(T1) $g \triangleright v = v$ *for all* $v \in V$.

(T2) $g \triangleright v = -v$ *for all* $v \in V$.

(T3) $g \triangleright v = \zeta_3 v$ *for all* $v \in V$ *and* $\dim(V) = 2$, *where* ζ_3 *is a 3-rd primitive root of unity.*

Nichols algebras of minimal nondiagonal module: first result

Remark

Let $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ be a minimal nondiagonal object such that $G = G_V$. Then V is a direct sum of 3 simple objects of the same dimension. So $\dim(V) = 3n$ with $n \geq 2$. (n=1 implies that V is diagonal type).

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Proposition

Let $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ be a minimal nondiagonal object such that $G = G_V$. If $\dim(V) \geq 9$, then $\mathrm{GKdim}(B(V)) = \infty$.

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A sketched proof:

(1) Let $V = V_1 \oplus V_2 \oplus V_3$ be a direct sum of simple objects, and V_1, V_2, V_3 are of types T1-T2 (type T3 implies the dimension of simple object must be 2). Then $B(V_1 \oplus V_2), B(V_1 \oplus V_3), B(V_2 \oplus V_3)$ are all of diagonal type.

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(2) By arithmetic root systems, we can prove that at least one of the three Nichols algebras is infinite GK-dimensional. So $\mathrm{GKdim}(B(V))$ is infinite.

One unresolved case

If $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ is a minimal nondiagonal object with $\dim(V) = 6$. We can prove $\mathrm{GKdim}(B(V)) = \infty$ by a similar method except the one case:

Example

Let $V = V_1 \oplus V_2 \oplus V_3 \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ be a minimal nondiagonal object with $G = G_V$ such that:

(1) $\dim(V) = 6$, or equivalently $\dim(V_1) = \dim(V_2) = \dim(V_3) = 2$;

(2) V_1, V_2, V_3 are all of type (T2).

By arithmetic root systems, we can show that $B(V_1 \oplus V_2)$, $B(V_1 \oplus V_3)$ and $B(V_2 \oplus V_3)$ are possible all finite GKdimensional.

Definition

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Let $P(V)$ be a pre-Nichols algebra of rank n, i.e., $V = \oplus_{1\leq i\leq n} V_i \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^n$ is a direct sum of n simple objects. Let $\{e_1, e_2, \ldots, e_n\}$ be a set of free generators of $\mathbb{Z}^n$. A pre-Nichols algebra $P(V)$ is called graded if it is a $\mathbb{Z}^n$-graded braided Hopf algebra such that $\deg(V_i) = e_i$ for $1 \leq i \leq n$.

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Fact

Every Nichols algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^\Phi$ is graded.

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Fact

Every Nichols algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^\Phi$ is graded.

Notation: Let $P(V)$ be a graded pre-Nichols algebra of rank n, and let $S \subset \mathcal{P}(V)$ be a homogenous subobject. Then we denote $D(S)$ the set of Z^n-degrees of nonzero homogenous elements in S.

A key observation

Proposition

Let $P(V)$ be a graded pre-Nichols algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ with counit ϵ, and let A be a homogenous subalgebra of $P(V)$. If there are homogenous subobjects $S, T \subset \ker \epsilon$ such that:

(a) $\Delta(A) \subset A \otimes A + S \otimes T$,

(b) $D(A) \cap D(S)$ *and* $D(A) \cap D(T)$ *are empty sets,*

(c) $S^2 \subset S, T^2 \subset T, ASA \subset S, ATA \subset T$,

then A is a braided bialgebra with a new coproduct $\widetilde{\Delta} = \pi \circ \Delta$, where

$$\pi : A \otimes A + S \otimes T \to A \otimes A$$

is the canonical projection.

A useful corollary

Corollary

Let $V = V_1 \oplus \cdots \oplus V_n \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ *be a direct sum of simple objects with* $n \geq 3$*, and* $\mathcal{P}(V)$ *a graded pre-Nichols algebra. Let*

$$W = \mathrm{ad}_{V_i}(\mathrm{ad}_{V_j}(V_k)) + \mathrm{ad}_{V_j}(\mathrm{ad}_{V_i}(V_k)) \subset \mathcal{P}(V)$$

with $i \neq j \neq k \in \{1, 2, \cdots, n\}$*, and let* $A(W)$ *be the subalgebra of* $\mathcal{P}(V)$ *generated by* W *. If* W *is nonzero, then* $A(W)$ *is a pre-Nichols algebra of* W *(with a new coproduct* $\widetilde{\Delta}$*), and thus* $B(W)$ *is a subquotient of* $P(V)$*.*

Proof: $A(W)$ satisfies all the conditions of previous proposition, and each nonzero element in W is primitive under new coproduct $\widetilde{\Delta}$.

Back to the unresolved case

Proposition

Let $V = V_1 \oplus V_2 \oplus V_3 \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ *be a minimal nondiagonal object and* $G = G_V$ *such that:*

(a) $\dim(V) = 6$;

(b) V_1, V_2, V_3 *are all of type (T2).*

Then $\mathrm{GKdim}(B(V)) = \infty$.

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Proof:

(1) Let $W = \mathrm{ad}_{V_1}(\mathrm{ad}_{V_2}(V_3)) + \mathrm{ad}_{V_2}(\mathrm{ad}_{V_1}(V_3))$. Then it is nonzero and is of diagonal. Using arithmetic root system we can prove

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$$\mathrm{GKdim}(B(W)) = \infty.$$

(2) Since $B(W)$ is a subquotient of $B(V)$, thus

$$\mathrm{GKdim}(B(V)) \geq \mathrm{GKdim}(B(W)) = \infty.$$

The main result

Theorem

Suppose $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ *is nondiagonal, then* $\mathrm{GKdim}(B(V)) = \infty$.

Proof: Let U be a minimal nondiagonal object insides V. Then $\mathrm{GKdim}(B(U)) = \infty$, and thus $\mathrm{GKdim}(B(V)) = \infty$.

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Corollary

Let $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ *be a nondiagonal object and let* $P(V)$ *be a pre-Nichols algebra of* V. *Then* $\mathrm{GKdim}(P(V)) = \infty$

Proof: there is a surjective homomorphism: $P(V) \twoheadrightarrow B(V)$.

The main result

A summary of finite GKdimensional Nichols algebras in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$:

B(V): diagonal type with finite GKdim $\longleftrightarrow$ arithmetic root systems

B(V): nondiagonal type $\longrightarrow$ infinite GKdimensional

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Theorem

Let $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^{\Phi}$ be a finite dimensional object. Then $\mathrm{GKdim}(B(V)) < \infty$ if and only if $B(V)$ is diagonal type and its root system is finite.

Thank You!