Frobenius-Schur indicators from invariants of 3-manifolds

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Frobenius-Schur indicators for finite groups

- Let G be a finite group. Consider the group algebra $\mathbb{C}G$.

- For any $\mathbb{C}G$-module V or $V \in \text{Rep}(G)$, χ_V denotes the character of G afforded by V.

- The n-th Frobenius-Schur indicator $\nu_n(V)$ of V is the scalar:

$$\nu_n(V) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g^n) = \chi_V\left(\frac{1}{|G|} \sum_{g \in G} g^n\right).$$

Counting with Frobenius-Schur indicators

- Let $\hat{G}$ denote the set of isomorphism classes of simple $\mathbb{C}G$-modules.
- [Frobenius-Schur] Let M be a compact non-orientable surface without boundary with Euler characteristic $\chi(M)$. Then

$$|\operatorname{Hom}(\pi_1(M), G)| = \frac{1}{|G|^{\chi(M)-1}} \sum_{V \in \hat{G}} (\nu_2(V) \dim(V))^{\chi(M)},$$

- and $\nu_2(V) \in \{0, 1, -1\}$ for $V \in \hat{G}$.
- What does $\nu_n(V)$ count?
- Let $\theta_n(g) = \#\{x \in G \mid x^n = g\}$
- Then θ_n is a class function.
- $$\theta_n(g) = \sum_{V \in \hat{G}} \nu_n(V)\chi_V(g) \quad \text{or} \quad \theta_n = \sum_{V \in \hat{G}} \nu_n(V)\chi_V \,.$$

FS-indicators beyond finite groups

- Rep(G) is pivotal fusion category.
- FS-indicators has been generalized to the setting of pivotal categories.
- They have been essential to the proofs congruence conjectures
  (i) in modular tensor categories (2010) and
  (ii) vertex operator algebras (2015),
  (iii) the rank finiteness conjecture of MTC by Zhenghan Wang (2016),
  (iv) a conjecture of Galois action on modular tensor categories by Coste-Gannon (2015), which lead to the development of higher Gauss sums and central charges and an answer to the conjecture of Davydov-Nikshych-Ostrik on Witt group (2022).
- Wishful Thinking: the development of such tools in the nonsemisimple settings is indispensable.

FS-indicators for semisimple Hopf algebras

- Let H be a finite-dimensional semisimple Hopf algebra over $\mathbb{C}$ with comultiplication $\Delta : H \to H \otimes H$ and counit $\varepsilon : H \to \mathbb{C}$ and antipode S.
- *Semisimple* Hopf algebras are generalizations of the group algebras $\mathbb{C}G$.
- [Sweedler] If H is a semisimple Hopf algebra, there exists a unique element $\Lambda \in H$ such that $x\Lambda = \varepsilon(x)\Lambda$ and $\varepsilon(\Lambda) = 1$.
- Power map can be defined for any element $x \in H$: $x^{[n]} := m(\Delta_n(x)) = x_{(1)}x_{(2)} \cdots x_{(n)}$.
- Let $\chi_V : H \to \mathbb{C}$ denote the character of an H-module V.
- [Linchenko-Montgomery] Let H be a semisimple Hopf algebra. The n-th Frobenius-Schur indicator of an H-module V is defined as

$$\boxed{\nu_n(V) := \chi_V(\Lambda^{[n]})}.$$

Frobenius-Schur Theorem and higher indicators

- [Linchenko-Montgomery] Let H be a semisimple Hopf algebra. For any simple H-module V,

$$\nu_2(V) \in \{0, 1, -1\}.$$

The value of $\nu_2(V)$ can be determined by the existence of certain H-invariant non-degenerate bilinear forms on V.

- The regular H-module is H considered as an H-module by the multiplication.
- [Kashina-Sommerhauser-Zhu] $\boxed{\nu_n(H) = \mathrm{Tr}(S \circ P_{n-1})}$, where $P_k(x) = x^{[k]}$ is the power map.
- $\nu_2(H) = \mathrm{Tr}(S)$, $\nu_1(H) = 1$ and $\nu_0(H) = \mathrm{Tr}(S^2) = \dim(H)$.

Invariance of these indicators

- [Ng-Schauenburg] Let H and K be quasi-Hopf Hopf algebras such that Rep(H) and Rep(K) are equivalent tensor category. Then there is a tensor equivalence $F : \text{Rep}(H) \to \text{Rep}(K)$ such that $F(H) \cong K$.
- If H and K are semisimple, and $\text{Rep}(H) \overset{\otimes}{\cong} \text{Rep}(K)$, then $\nu_n(H) = \nu_n(K)$ for all integers n.
- $\nu_2(\mathbb{C}D_8) = 6$ and $\nu_2(\mathbb{C}Q_8) = 2 \quad \Rightarrow \quad \text{Rep}(D_8) \overset{\otimes}{\ncong} \text{Rep}(Q_8)$.
- A quantity $f(H)$ defined for each member H in a family of Hopf algebras, whose representation categories are closed under tensor equivalence, is called a gauge invariant if $f(H) = f(K)$ whenever Rep(H) and Rep(K) are equivalent tensor categories.
- The family of involutory Hopf algebras ($S^2 = \text{id}$) over a field of characteristic zero is closed (Larson-Radford).
- However, this family in positive characteristic is NOT closed.

Higher indicators of non-semisimple Hopf algebras

- In any non-semisimple Hopf algebras H, Left (resp. right) integrals $\{\Lambda \mid x\Lambda = \varepsilon(x)\Lambda\}$ form a 1-dimensional subspace of H but $\varepsilon(\Lambda) = 0$.
- The expression $\nu_n^{KMN}(H) := \mathrm{Tr}(S \circ P_{n-1})$ is well-defined for any finite-dimensional Hopf algebras over any field.
- [Kashina-Montgomery-Ng] The sequence $\{\nu_n^{KMN}(H)\}$ is a gauge invariant of any finite dimensional Hopf algebra H over any field, i.e. if H' is another Hopf algebra such that H'-mod is equivalent to H-mod, then $\{\nu_n^{KMN}(H)\} = \{\nu_n^{KMN}(H)'\}$.
- By computing $\nu_2^{KMN}(T)$ for any Taft algebra T, the module categories of any two non-isomorphic Taft algebras are inequivalent as tensor categories.
- If $H = T_4(-1)$, then $\nu_n(H) = n$.
- [Shimizu] If $H = u_q\mathfrak{sl}_2$, with $q = e^{2\pi i/3}$, then $\nu_n(H) = n^2$.

Kuperberg invariants of 3-manifolds

- Fusion categories are sources of invariants of 3-manifolds via topological quantum field theory (TQFT).
- If H is semisimple, $\nu_n(H)/dim(H)$ is the RT-invariant of $L(n,1)$ from the MTC Rep$(D(H))$.
- [Question:] Are these sequence of scalars $\{\nu_n^{KMN}(H)\}$ invariants of some 3-manifolds?
- Given any finite-dimensional Hopf algebra H over any field,
- Kuperberg constructed an invariant $K(M, f, H)$ for any closed oriented 3-manifold M with a framing f.
- [Chang-Z. Wang] If H is a *factorizable, ribbon* Hopf algebra over $\mathbb{C}$, then the Kuperberg invariant $K(M, f, H)$ of a Lens space M is independent of its framing f. Moreover,

$$K(L(n,1), f, H) = \nu_n^{KMN}(H).$$

Kuperberg invariants of some Lens spaces

- [Chang-Ng-Y. Wang] For any Hopf algebra H, there is a specific framing f_n of $L(n,1)$ for each positive integer n such that

$$K(L(n,1), f_n, H) = \nu_n^{KMN}(H).$$

- The right hand side was known to be a gauge invariant, and so is this Kuperberg invariant.

- [Question] Is the Kuperberg invariant $K(M, f, H)$ for any framed 3-manifold (M, f) and Hopf algebra H a gauge invariant of H?

- We have addressed this question for all genus 1 and a family of genus 2 framed 3-manifolds in our preprint arxiv:2506.07409.

Kuperberg invariants of framed 3-manifolds of genus 1

- Recall the Lens space $L(n,k)$ for any coprime integers n, k is the quotient space of $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ by the order n cyclic group $A_n = \langle g \rangle$ with the action:

$$g \cdot (z_1, z_2) = (e^{2\pi i/n} z_1, e^{2k\pi i/n} z_2)\,.$$

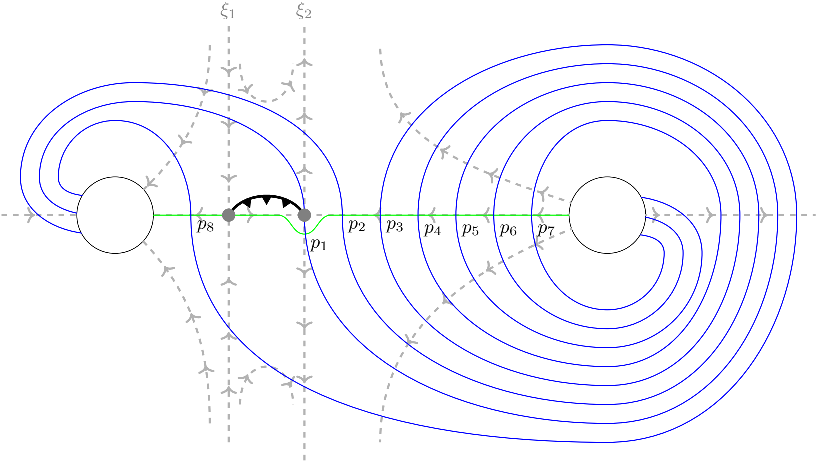
- What about the Kuperberg invariant $K(L(n,k), f, H)$?

- [Chang-Ng-Y. Wang] For any coprime $n, k \in \mathbb{N}$, there is a framing f on $L(n,k)$ s.t. $\boxed{K(L(n,k), f, H) = \mathrm{Tr}(S \circ P_{n,k})}$ for some linear map $P_{n,k} \in \mathrm{End}_{\mathbb{C}}(H)$ which is well-defined for any f.d Hopf algebra H.
  Moreover, $K(L(n,k), f, H)$ is a gauge invariant for any framing f on $L(n,k)$.

Construction of Kuperberg invariants from a framed 3-manifolds

- A closed oriented 3-manifold M can always be obtained by gluing 2 genus n handle bodies on their boundaries Σ.
- The way these handle bodies are glued together is based on a mapping class of Σ, or a homeomorphism of Σ up to homotopy.
- A mapping class can be identified with n-disjoint closed curves on Σ.
- A framing of a 3-manifold consists of 3 1-dimension vector bundles which are orthogonal.
- The first vector bundle restricted on Σ is presented as a tangent bundle.
- The singularities will be the based points of these curves.
- Σ is presented on the plane with pairs of holes which represent the handles above the plane.
- The twist fronts are curves between based points on which the the second vector bundle is point out the plane.
- The 3rd vector bundle is determined by the right hand rule.

The Heegaard diagram of $L(8,3)$ with a framing f.



Combinatorial data of the Heegaard diagram

The rotation numbers of the framed Heegaard diagram of f are recorded in the following table.

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	total
θ_η	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
θ_μ	0	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$	$-\frac{1}{2}$
S-term	S	S^3	S	S	S^3	S	S	S^3	
ϕ_η	0	0	0	0	0	0	0	0	$\frac{1}{2}$
ϕ_μ	0	0	0	0	0	0	0	0	$\frac{1}{2}$
T-term	id	id	id	id	id	id	id	id	

.

$$K(L(8,3),f,H) = (g \rightharpoonup \lambda)\Big(S(\Lambda_{(1)})S(\Lambda_{(4)})S(\Lambda_{(7)})S^3(\Lambda_{(2)})S^3(\Lambda_{(5)}) S^3(\Lambda_{(8)})S(\Lambda_{(3)})S(\Lambda_{(6)})\Big)$$

$$=\lambda\Big(\Lambda_{(6)}\Lambda_{(3)}S^2(\Lambda_{(8)})S^2(\Lambda_{(5)})S^2(\Lambda_{(2)})\Lambda_{(7)}\Lambda_{(4)}\Lambda_{(1)}\Big).$$

where $\lambda \in H^*$ is a right integral and $\Lambda \in H$ a left integral such that $\lambda(\Lambda) = 1$.

Main Theorem

- Using the Radford trace formula, one can obtain

$$K(L(n,k), f, H) = \mathrm{Tr}(S \circ P_{n,k})$$

  for any coprime integers $n > k > 0$.
- $P_{n,n-1}(x) = P_{n-1}(x) = x^{[n-1]}$
- $P_{5,2}(x) = \sum_{(x)} x_{(3)} S^{-2}(x_{(1)}) S^{-2}(x_{(4)}) x_{(2)}$.

Theorem (Chang-Ng-Y. Wang)

The Kuperberg invariant $K(L(n,k), f, H)$ for any framing f of $L(n,k)$ is a gauge invariant.

- Let $F \in H \otimes H$ be a 2-cocycle, and H_F the corresponding Drinfeld twist of H.

$$K(L(n,k), f, H) = K(L(n,k), f, H_F)\,.$$

- $K(S^3, f, H) = 1$ for any Hopf algebra.
- $K(S^2 \times S^1, f, H) = \mathrm{Tr}(S^2)$ for any Hopf algebra.

Kuperberg invariants of the orbifolds of $SU(2)$

- How far can we push this idea?
- $SU(2) = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \mid |a|^2 + |b|^2 = 1 \right\} \simeq S^3.$
- Consider $A_n = \left\langle \begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix} \right\rangle \subset SU(2).$
- $\mathrm{SU}(2)/A_n \simeq L(n, n-1) \simeq L(n, 1).$
- $\nu_n^{KMN}(H) = K(\mathrm{SU}(2)/A_n, f, H)$ for some framing f.
- The dicyclic group Dic_n of order $4n$ for $n \geq 2$, a central $\mathbb{Z}_2$ extension of D_{2n}, is also a subgroup of $SU(2)$.
- $K(SU(2)/Dic_n, f_n, H)$ is another sequence of 3-manifold invariants.
- [Chang-Ng-Y. Wang] For each positive integer n, there exists a framing f_n such that $K(SU(2)/Dic_n, f_n, H)$ is a gauge invariant.

Kuperberg invariants of Q_8-orbifolds

- $Dic_2 = Q_8$.
- There exists a framing f on $SU(2)/Q_8$ such that
- $K(SU(2)/Q_8, f, H) =$

$$\lambda(\Lambda^2_{(4)} S(\Lambda^1_{(3)}) S^2(\Lambda^2_{(2)}) \Lambda^1_{(1)}) \cdot \lambda(S^2(\Lambda^1_{(4)}) S(\Lambda^2_{(3)}) \Lambda^1_{(2)} \Lambda^2_{(1)}),$$

where $\lambda \in H^*$ is a right integral and $\Lambda^1 \otimes \Lambda^2 \in H \otimes H$ is left integral such that $\lambda(\Lambda^1) = \lambda(\Lambda^2) = 1$.

Heegaard diagram of S^3/Q_8

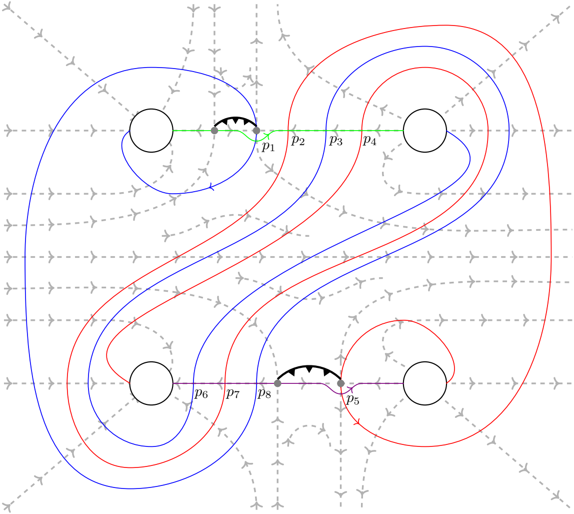


Figure 1: A framed Heegaard diagram of $M = S^3/Q_8$.

Thank you for your attention !