Casimir and polynomial invariants of Hopf algebras

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• Classify: $\{\text{groups}\} \longrightarrow \{\text{Hopf algebras}\} \longrightarrow \{\text{tensor categories}\}$.

• Aim: to understand two important invariants of Hopf algebras : Casimir and polynomial invariants.

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Theorem

$\Bbbk \mid X^* \otimes X \Leftrightarrow X \otimes E \xrightarrow{id_X \otimes \sigma} X$, *right almost split hom (asked by Cibils, Comm. Phy.Math, 1993).*

- Bilinear form on $r(H)$

Define a bilinear form on $r(H)$: for any $X, Y \in Rep(H)$,

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- if S^2 is inner, then $(\ ,\ )$ is symmetric;
- $r(H)/\mathcal{P}^{\perp} \cong g(H)$, P : proj ideal generated by proj modules.

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(1) $\{[Z], \delta^*_{[Z]} \mid [Z] \in ind(H)\}$: *dual basis of* $r(H)$;.
(2) $[Z] = \sum_{[M]\in ind(H)} \dim_k Hom_H(Z, M^*)\delta^*_{[M]}$;
(3) if H : *finite rep type, then* $r(H)$ *is a Frobenius algebra over* $\mathbb{Z}$.

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Moreover we have

- $\Bbbk \nmid X^* \otimes X \Leftrightarrow [X]\delta_{[\Bbbk]} = 0$.

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I. Casimir number of a f.d Hopf algebra

- H : a f.d. Hopf algebra of finite representation type. Let $r(H)$ be the Green ring of H with a pair of dual pases $\{\delta^*_{[X]}, [X] \mid [X] \in \mathrm{ind}(H)\}$ with respective to the bilinear form $([X],[Y]) = \dim_{\Bbbk} \mathrm{Hom}_H(X, Y^*)$. Consider the Casimir operator

$$c : r(H) \to r(H), c(x) = \sum_{[X]\in \mathrm{ind}(H)} [X] x \delta^*_{[X]}.$$

- $(m_H) = Imc \cap \mathbb{Z}$ is an ideal of $\mathbb{Z}$, m_H is called the Casimir number of $\mathrm{Rep}(H)$.
- $d_H = det([c(1)])$, d_H is called the determinant of $\mathrm{Rep}(H)$.

Examples

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- $m_{\mathbb{C}S_5} = 7200 = 2^5 \cdot 3^2 \cdot 5^2$, $d_{\mathbb{C}S_5} = 8294400 = 2^{12} \cdot 3^4 \cdot 5^2$
- $m_{\Bbbk G} = 2p^2 (char \Bbbk = p,\ G\colon$ cyc with order p)

Maschke theorem

Theorem

Let H be a Hopf algebra of finite representation type over $\mathbb{k}$. The Green algebra $r(H) \otimes_{\mathbb{Z}} F$ over a field F is Jacobson semisimple iff the Casimir number m_H of $Rep(H)$ is not zero in F.

Remark: If $p \mid dim_{\mathbb{k}}(H)$, then $r(H) \otimes_{\mathbb{Z}} F$ is not Jacobson semisimple for a field F of $charF = p$.

Theorem

Let H be a Hopf algebra of finite representation type over $\mathbb{k}$. The Green ring $r(H)$ is Jacobson semisimple iff the Casimir number of $Rep(H)$ is not zero.

Example: Non-semisimple

Let G be a cyclic group of order p and the field $\Bbbk$ has characteristic p. Consider the Hopf algebra $\Bbbk G$:

- The Green ring $r(\Bbbk G) \cong \mathbb{Z}[X]/((X-2)E_{p-1}(X))$;
- The Casimir number of $r(\Bbbk G)$ is $2p^2$;
- The Green ring $\mathbb{Z}[X]/((X-2)E_{p-1}(X))$ is semisimple;
- The Green algebra $F[X]/((X-2)E_{p-1}(X))$ is semisimple iff $charF \neq 2, p$;
- If $charF = p$, then the Jacobson radical of $F[X]/((X-2)E_{p-1}(X))$ is generator by $\overline{X^2-4}$;
- If $charF = 2$, then the Jacobson radical of $F[X]/((X-2)E_{p-1}(X))$ is generator by

$$\sum_{i=0}^{[\frac{p-1}{2}]} \binom{p-1-i}{i} (-1)^i \overline{X^{\frac{p+1}{2}-i}}.$$

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For a pivotal semisimple Hopf algebra H,

- $Rep(H)$ is non-degenerate $\Leftrightarrow g(H) \otimes_{\mathbb{Z}} \Bbbk$ is semisimple $\Leftrightarrow m_H$ or $d_H \neq 0$ in $\Bbbk$.

Proposition

Let H be a semisimple and cosemisimple Hopf algebra over $\Bbbk$. If $Gr(H)$ is commutative, then $d_{Rep(H)}$ and $\dim_{\Bbbk} H$ have the same prime factors.

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Theorem

Let H be a semisimple and cosemisimple Hopf algebra over $\Bbbk$ and $D(H)$ the Drinfeld double of H. Then $d_{Rep(D(H))}$ and $\dim_{\Bbbk} H$ have the same prime factors.

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Question: For a semisimple Hopf algebra H, $p \mid \dim_{\Bbbk} H \Leftrightarrow p \mid d_{\mathrm{Rep}(H)}$?

II. Polynomial invariants of a f.d Hopf albegra

- The semisimple Hopf algebras of dimension 8 (over $\mathbb{C}$):
- ▷ The quaternion group $\mathbb{C}Q_8$
- ▷ the dihedral group $\mathbb{C}D_4$
- ▷ 8-dimensional Kac algebra K_8
- The non-isomorphic pointed non-semisimple HA of dimension 12 :

$$\mathcal{A}_0: \ g^6 = 1, \ x^2 = 0, \ gx = -xg, \ \ \Delta(x) = x \otimes 1 + g \otimes x,$$
$$\mathcal{A}_1: \ g^6 = 1, \ x^2 = 1 - g^2, \ gx = -xg, \ , \ \Delta(x) = x \otimes 1 + g \otimes x,$$
$$\mathcal{B}_0: \ g^6 = 1, \ x^2 = 0, \ gx = -xg, \ \ \Delta(x) = x \otimes 1 + g^3 \otimes x,$$
$$\mathcal{B}_1: \ g^6 = 1, \ x^2 = 0, \ gx = \omega xg, \ \ \Delta(x) = x \otimes 1 + g^3 \otimes x.$$

- $\Bbbk$ is a fixed field.
- H is a f.d. Hopf algebra over $\Bbbk$ with coproduct: $\Delta(a) = a_{(1)} \otimes a_{(2)}$.
- $\Lambda \in H$ is a left integral and $\lambda \in H^*$ is a right integral satisfying $\lambda(\Lambda) = 1$. α is the distinguished group-like element in H^* defined by $\Lambda a = \alpha(a)\Lambda$ for $a \in H$.
- The n-th Sweedler power map $P_n : H \to H$ is defined by

$$P_n(a) = \begin{cases} a_{(1)} \cdots a_{(n)}, & n \geq 1; \\ \varepsilon(a), & n = 0; \\ S(a_{(-n)} \cdots a_{(1)}), & n \leq -1. \end{cases}$$

Polynomial invariants of a f.d Hopf albegra

• A (normalized) twist for H is an invertible element $J = J^{(1)} \otimes J^{(2)} \in H \otimes H$ which satisfies

$$(\varepsilon \otimes id)(J) = (id \otimes \varepsilon)(J) = 1$$

and

$$(\Delta \otimes id)(J)(J \otimes 1) = (id \otimes \Delta)(J)(1 \otimes J).$$

• $H \overset{J}{\rightsquigarrow} H^J$, H^J has the same algebra structure and counit as H, for which the coproduct Δ^J and antipode S^J are given respectively by

$$\Delta^J(a) = J^{-1}\Delta(a)J, \qquad S^J(a) = Q_J^{-1}S(a)Q_J \text{ for } a \in H,$$

where $Q_J = S(J^{(1)})J^{(2)}$ and $Q_J^{-1} = J^{-(1)}S(J^{-(2)})$.

• Note that a nonzero left integral Λ of H is also a left integral of H^J.

• The n-th Sweedler power map of H^J is denoted by P_n^J.

Polynomial invariants of a f.d Hopf albegra

If $\mathcal{F} : \mathrm{Rep}(H) \to \mathrm{Rep}(H')$ is an equivalence of tensor categories, then there is a normalized twist J of H such that $H' \overset{\sigma}{\cong} H^J$ as bialgebras (σ is automatically a Hopf algebra isomorphism). Therefore,

$$\sigma \circ P'_n = P^J_n \circ \sigma, \quad n \in \mathbb{Z}.$$

Suppose Λ and Λ' are left integrals of H and H' respectively and $\sigma(\Lambda') = \mu\Lambda$ for a nonzero scalar $\mu \in \mathbb{k}$.

$$\sigma(P'_n(\Lambda')) = P^J_n(\sigma(\Lambda')) = \mu P^J_n(\Lambda) \leftrightsquigarrow P_n(\Lambda),$$

Question: What is the relationship between $P^J_n(\Lambda)$ and $P_n(\Lambda)$?

A relationship between $P_n^J(\Lambda)$ and $P_n(\Lambda)$

Theorem

Let H be a finite-dimensional Hopf algebra over the field $\Bbbk$ with a nonzero left integral Λ and a normalized twist $J = J^{(1)} \otimes J^{(2)}$. Then

$$P_n^J(\Lambda) = TP_n(\Lambda) \text{ for } n \in \mathbb{Z},$$

where $T = J^{-(1)}\alpha(J^{-(2)})$.
If H is unimodular, then $P_n^J(\Lambda) = P_n(\Lambda)$.

Remark: For a unimodular Hopf algebra H, the sequence $\{P_n(\Lambda)\}_{n\in\mathbb{Z}}$ is invariant under twisting.

Some applications of $\sigma(P'_n(\Lambda')) = \mu T P_n(\Lambda)$

For any $a \in H$, we define $a_k \in H$ recursively by $a_1 = a_{(1)}(\alpha \circ S^{-1})(a_{(2)})$ and $a_{k+1} = (a_k)_{(1)}(\alpha \circ S^{-1})((a_k)_{(2)})$ for $k \geq 1$.

• $a(P_n(\Lambda))^k = (P_n(\Lambda))^k a_k$ for $n \in \mathbb{Z}$.

•$(aP_n(\Lambda))^k = (P_n(\Lambda))^k a_k \cdots a_2 a_1$, $n \in \mathbb{Z}$.

• For $k_1, \cdots, k_s \in \mathbb{Z}^{>0}$, $n_1, \cdots, n_s \in \mathbb{Z}$,

$$(aP_{n_1}(\Lambda))^{k_1} \cdots (aP_{n_s}(\Lambda))^{k_s} = (P_{n_1}(\Lambda))^{k_1} \cdots (P_{n_s}(\Lambda))^{k_s} a_{k_1+\cdots+k_s} \cdots a_2 a_1.$$

Some applications of $\sigma(P'_n(\Lambda')) = \mu T P_n(\Lambda)$

- Let $\psi(X_1, \cdots, X_s)$ be a homogeneous polynomial of degree m and $H' \overset{\sigma}{\cong} H^J$. Then

$$\begin{aligned}
\sigma(\psi(P'_{n_1}(\Lambda'), \cdots, P'_{n_s}(\Lambda'))) &= \psi(\sigma(P'_{n_1}(\Lambda')), \cdots, \sigma(P'_{n_s}(\Lambda'))) \\
&= \psi(\mu T P_{n_1}(\Lambda), \cdots, \mu T P_{n_s}(\Lambda)) \\
&= \psi(P_{n_1}(\Lambda), \cdots, P_{n_s}(\Lambda))\mu^m T_m \cdots T_2 T_1.
\end{aligned}$$

Some applications of $\sigma(P'_n(\Lambda')) = \mu T P_n(\Lambda)$

Theorem

Let H and H' be gauge equivalent finite-dimensional Hopf algebras over the field $\Bbbk$ with nonzero left integrals Λ and Λ' respectively. For any homogeneous polynomial $\psi(X_1, \cdots, X_s) \in \Bbbk[X_1, \cdots, X_s]$, we have $\psi(P_{n_1}(\Lambda), \cdots, P_{n_s}(\Lambda)) = 0$ if and only if $\psi(P'_{n_1}(\Lambda'), \cdots, P'_{n_s}(\Lambda')) = 0$.

Remark: Any homogeneous polynomial $\psi(X_1, \cdots, X_s) \in \Bbbk[X_1, \cdots, X_s]$ satisfying $\psi(P_{n_1}(\Lambda), \cdots, P_{n_s}(\Lambda)) = 0$ is an invariant of $\mathrm{Rep}(A)$.

Example

The following is a list of pairwise nonisomorphic pointed nonsemisimple Hopf algebras of dimension 12 over an algebraically closed field $\Bbbk$ of characteristic zero.

$$\mathcal{A}_0: \quad g^6 = 1,\ x^2 = 0,\ gx = -xg,\ \Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + g \otimes x,$$
$$\mathcal{A}_1: \quad g^6 = 1,\ x^2 = 1 - g^2,\ gx = -xg,\ \Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + g \otimes x,$$
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These Hopf algebras have nonzero left integrals of the same form

$$\Lambda = (1 + g + g^2 + g^3 + g^4 + g^5)x.$$

$\mathcal{A}_0$	$P_2(\Lambda) + P_{-2}(\Lambda) = 0$	$P_3(\Lambda) - 3P_2(\Lambda) - P_{-3}(\Lambda) = 0$
$\mathcal{A}_1$	$P_2(\Lambda) + P_{-2}(\Lambda) = 0$	$P_3(\Lambda) - 3P_2(\Lambda) - P_{-3}(\Lambda) = 0$
$\mathcal{B}_0$	$P_2(\Lambda) + P_{-2}(\Lambda) = 0$	$P_3(\Lambda) - 3P_2(\Lambda) - P_{-3}(\Lambda) \neq 0$
$\mathcal{B}_1$	$P_2(\Lambda) + P_{-2}(\Lambda) \neq 0$	

Conclusion: The representation categories of these Hopf algebras are mutually inequivalent as tensor categories.

Some applications of $\sigma(P'_n(\Lambda')) = \mu T P_n(\Lambda)$

If H is unimodular, then $T = J^{-(1)}\alpha(J^{-(2)}) = 1$ and hence $\sigma(P'_n(\mu^{-1}\Lambda')) = P_n(\Lambda)$. Let $\psi(X_1, \cdots, X_s)$ be a polynomial and $H' \overset{\sigma}{\cong} H^J$.

$$\begin{aligned}\sigma(\psi(P'_{n_1}(\mu^{-1}\Lambda'), \cdots, P'_{n_s}(\mu^{-1}\Lambda'))) &= \psi(\sigma(P'_{n_1}(\mu^{-1}\Lambda')), \cdots, \sigma(P'_{n_s}(\mu^{-1}\Lambda'))) \\ &= \psi(P_{n_1}(\Lambda), \cdots, P_{n_s}(\Lambda)).\end{aligned}$$

Theorem

Let H and H' be gauge equivalent semisimple Hopf algebras with idempotent integrals Λ and Λ' respectively. For any polynomial $\psi(X_1, \cdots, X_s)$, $\psi(P_{n_1}(\Lambda), \cdots, P_{n_s}(\Lambda)) = 0$ if and only if $\psi(P'_{n_1}(\Lambda'), \cdots, P'_{n_s}(\Lambda')) = 0$.

Remark: If H is semisimple with an idempotent integral Λ, then any polynomial $\psi(X_1, \cdots, X_s)$ satisfying $\psi(P_{n_1}(\Lambda), \cdots, P_{n_s}(\Lambda)) = 0$ is an invariant of $\mathrm{Rep}(H)$.

Example

The quaternion group Q_8 is a group with eight elements: $\mathbb{1}, -\mathbb{1}, i, -i, j, -j, k, -k$, where $\mathbb{1}$ is the identity element, $(-\mathbb{1})^2 = \mathbb{1}$ and all the other elements are squareroots of $-\mathbb{1}$, such that $(-\mathbb{1})i = -i, (-\mathbb{1})j = -j, (-\mathbb{1})k = -k$ and further, $ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j$ (the remaining relations can be deduced from these). The group algebra $\Bbbk Q_8$ has the idempotent integral

$$\Lambda = \frac{1}{8}(\mathbb{1} + (-\mathbb{1}) + i + (-i) + j + (-j) + k + (-k)).$$

By a straightforward computation, we have

$$P_1(\Lambda) = \Lambda, \quad P_2(\Lambda) = \frac{1}{4}\mathbb{1} + \frac{3}{4}(-\mathbb{1}), \quad P_3(\Lambda) = P_1(\Lambda), \quad P_4(\Lambda) = P_0(\Lambda).$$

Note that $P_4(\Lambda) - P_0(\Lambda) = 0$ and $2(P_2(\Lambda))^2 - P_2(\Lambda) - P_0(\Lambda) = 0$.

Example

For the dihedral group $D_4 = \{1, a, a^2, a^3, b, ba, ba^2, ba^3\}$ with $a^4 = 1,\ b^2 = 1$ and $aba = b$, the group algebra $\Bbbk D_4$ has the idempotent integral

$$\Lambda = \frac{1}{8}(1 + a + a^2 + a^3 + b + ba + ba^2 + ba^3).$$

We have

$$P_1(\Lambda) = \Lambda, \quad P_2(\Lambda) = \frac{3}{4} + \frac{1}{4}a^2, \quad P_3(\Lambda) = P_1(\Lambda), \quad P_4(\Lambda) = P_0(\Lambda).$$

Note that $P_4(\Lambda) - P_0(\Lambda) = 0$ while $2(P_2(\Lambda))^2 - P_2(\Lambda) - P_0(\Lambda) = \frac{1}{2}a^2 - \frac{1}{2} \neq 0$.

Example

The 8-dimensional Kac algebra K_8 is a semisimple Hopf algebra over $\Bbbk$ generated by x, y, z as a $\Bbbk$-algebra with the following relations:

$$x^2 = y^2 = 1, \; z^2 = \frac{1}{2}(1 + x + y - xy), \; xy = yx, \; xz = zy, \; yz = zx.$$

The coalgebra structure Δ, ε and the antipode S of K_8 are given by

$$\Delta(x) = x \otimes x, \; \Delta(y) = y \otimes y, \; \varepsilon(x) = \varepsilon(y) = 1,$$

$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z), \; \varepsilon(z) = 1,$$

$$S(x) = x, \; S(y) = y, \; S(z) = z.$$

The idempotent integral of K_8 is

$$\Lambda = \frac{1}{8}(1 + x + y + xy) + \frac{1}{8}(1 + x + y + xy)z.$$

A straightforward computation shows that

$$P_1(\Lambda) = \Lambda, \quad P_2(\Lambda) = \frac{3}{4} + \frac{1}{4}xy, \quad P_3(\Lambda) = P_1(\Lambda), \quad P_4(\Lambda) = \frac{1}{2} + \frac{1}{2}xy,$$

$$P_5(\Lambda) = P_1(\Lambda), \quad P_6(\Lambda) = P_2(\Lambda), \quad P_7(\Lambda) = P_1(\Lambda), \quad P_8(\Lambda) = P_0(\Lambda).$$

Note that $P_4(\Lambda) - P_0(\Lambda) = \frac{1}{2}xy - \frac{1}{2} \neq 0$.

Example

We collect these relations in the following table:

K_8	$P_4(\Lambda) - P_0(\Lambda) \neq 0$		$P_4(\Lambda) - P_0(\Lambda) \neq 0$
$\Bbbk D_4$	$P_4(\Lambda) - P_0(\Lambda) = 0$	$2(P_2(\Lambda))^2 - P_2(\Lambda) - P_0(\Lambda) \neq 0$	
$\Bbbk Q_8$		$2(P_2(\Lambda))^2 - P_2(\Lambda) - P_0(\Lambda) = 0$	$P_4(\Lambda) - P_0(\Lambda) = 0$

Conclusion: The representation categories Rep(K_8), Rep($\Bbbk D_4$) and Rep($\Bbbk Q_8$) are mutually inequivalent as tensor categories.

A relationship between $\lambda \in H^*$ and $\lambda^J \in (H^J)^*$

For a unimodular Hopf algebra H, a right integral $\lambda^J \in (H^J)^*$ has been described by E.Aljadeff, P.Etingof, S.Gelaki, D.Nikshych(2002):

$$\lambda^J = \lambda \leftharpoonup S(Q_J^{-1})Q_J.$$

For a general Hopf algebra H, a right integral $\lambda^J \in (H^J)^*$ can be described as follows:

Theorem

Let A be a finite-dimensional Hopf algebra over the field $\mathbb{k}$ with a normalized twist J. Let $R = \alpha(J^{-(1)})J^{-(2)}$. If λ is a nonzero right integral in H^*, then

$$\lambda^J := \lambda \leftharpoonup S^2(R^{-1})S(Q_J^{-1})Q_J$$

is a nonzero right integral in $(H^J)^*$.

A uniform proof of the invariance of $\nu_n(H)$

- $\nu_n(H^J) = \nu_n(H)(n \geq 1$,Kashina-Montgomery-Ng,2012)
- $\nu_n(H^J) = \nu_n(H)$ ($n \leq 0$, Shimizu, 2015)
- **A uniform proof:** Note that the n-th indicator

$$\nu_n(H) = tr(S \circ P_{n-1}) = (\lambda \circ S)(P_n(\Lambda)),$$

where $\lambda(\Lambda) = 1$ and $n \in \mathbb{Z}$.

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$$\nu_n(H) = tr(S \circ P_{n-1}) = (\lambda \circ S)(P_n(\Lambda)),$$

where $\lambda(\Lambda) = 1$ and $n \in \mathbb{Z}$. Using $\lambda^J = \lambda \leftharpoonup S^2(R^{-1})S(Q_J^{-1})Q_J$ and $P_n^J(\Lambda) = TP_n(\Lambda)$, we may prove

$$(\lambda \circ S)(P_n(\Lambda)) = (\lambda^J \circ S^J)(P_n^J(\Lambda)), \text{ namely, } \nu_n(H) = \nu_n(H^J).$$

The Killing form of a Hopf algebra

For $a \in H$, the (left) adjoint action ad_a is the map of H given by

$$(\mathrm{ad}_a)(b) = a_{(1)} b S(a_{(2)}) \text{ for } b \in H.$$

The Killing form of the Hopf algebra H is defined by

$$(a, b) = \mathrm{tr}(\mathrm{ad}_a \circ \mathrm{ad}_b) = \mathrm{tr}(\mathrm{ad}_{ab}) \text{ for } a, b \in H.$$

The subspace $A^{\perp} = \{a \in H \mid (a, b) = 0 \text{ for all } b \in H\}$ is called the Killing radical of H.

The Killing form of a Hopf algebra

Using $P_n^J(\Lambda) = TP_n(\Lambda)$, we may give an alternative proof of the following result:

Theorem

The Killing forms of H is invariant under twist. Namely, if $(-,-)$ and $(-,-)^J$ are the Killing forms of H and H^J respectively, then $(a,b) = (a,b)^J$ for all $a, b \in H$.

Conclusion: $H^{\perp} = (H^J)^{\perp}$ and hence $\dim_{\Bbbk} H^{\perp} = \dim_{\Bbbk}(H^J)^{\perp}$. Namely, $\dim_{\Bbbk} H^{\perp}$ is an invariant of $\mathrm{Rep}(H)$ as a tensor category.

Example

★ **The dihedral group algebra**

Consider the dihedral group

$$D_n = \{1, a, \cdots, a^{n-1}, b, ba, \cdots, ba^{n-1}\}$$

with $a^n = 1$, $b^2 = 1$ and $aba = b$. Suppose $2n \neq 0$ in $\Bbbk$.

- If n is odd, then $(\Bbbk D_n)^{\perp} = 0$.
- If n is even, then $(\Bbbk D_n)^{\perp} = \langle 1 - a^{\frac{n}{2}} \rangle$. This is a Hopf ideal of $\Bbbk D_n$, $\dim_{\Bbbk}(\Bbbk D_n)^{\perp} = n$ and the quotient is $\Bbbk D_n/(\Bbbk D_n)^{\perp} \cong \Bbbk D_{\frac{n}{2}}$.

Example

★ **The generalized Taft algebra**

• $H_{n,d}$ denote a generalized Taft algebra over the field $\mathbb{k}$ with $nd \neq 0$ in $\mathbb{k}$. As an algebra, $H_{n,d}$ is generated by g and h subject to the following relations:

$$g^n = 1,\ h^d = 0,\ hg = qgh,$$

where d divides n and q is a primitive d-th root of unity.

• As a Hopf algebra, the coproduct Δ, counit ε and the antipode S of $H_{n,d}$ are given respectively by

$$\Delta(g) = g \otimes g,\ \Delta(h) = 1 \otimes h + h \otimes g,\ \varepsilon(g) = 1,\ \varepsilon(h) = 0,$$

$$S(g) = g^{-1},\ S(h) = -q^{-1}g^{n-1}h.$$

Example

- The Hopf algebra $H_{n,d}$ has a $\Bbbk$-basis $\{g^i h^j \mid 0 \leq i \leq n-1, 0 \leq j \leq d-1\}$ and $\dim_{\Bbbk}(H_{n,d}) = nd$.

- If $d = n$, then $H_{n,n}$ is the Taft algebra.

- The Killing radical is $H_{n,d}^{\perp} = \langle g^d - 1, h \rangle$. The dimension $\dim_{\Bbbk}(H_{n,d}^{\perp}) = (n-1)d$, which is an invariant of $\mathrm{Rep}(H_{n,d})$.

Thank you!