Geometric constructions of multi-parameter quantum Schur algebras and iquantum groups

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Background I: Iwahori's construction of Hecke algebras

- $\mathbb{F}_{\mathbf{q}}$: finite field with $\mathbf{q} = p^a$ elements; p is a prime number.
- $\mathbf{G}$: a connected reductive algebraic group defined over $\mathbb{F}_{\mathbf{q}}$.
- $G = \mathbf{G}(\mathbb{F}_{\mathbf{q}})$: the group of $\mathbb{F}_{\mathbf{q}}$-points of $\mathbf{G}$.
- Fix a Borel subgroup B and a maximal torus $T \subset B$ of G.
- $W = N_G(T)/T$: the Weyl group of G (associated to T).
- $\mathscr{B} = G/B$: complete flag variety.
- G acts diagonally on $\mathscr{B} \times \mathscr{B} = \bigsqcup_{w \in W} \mathscr{O}_w$. The G-orbits $\mathscr{O}_w$ are in one-to-one correspondence with $w \in W$ thanks to the Bruhat decomposition $G = \bigsqcup_{w \in W} B\dot{w}B$.
- The convolution product $\star$ on the space of G-invariant functions on $\mathscr{B} \times \mathscr{B}$ is defined by

$$f \star h(B, \dot{w}B) = \sum_{xB \in G/B} f(B, xB) h(xB, \dot{w}B).$$

Background I: Iwahori's construction of Hecke algebras

- T_w: the characteristic function of $\mathscr{O}_w$ $(w \in W)$.
- $s_1, \ldots, s_d$: the simple reflections of W.

$$
\begin{aligned}
T_{s_i} \star T_{s_i}(B, \dot{w}B) &= \sum_{xB \in G/B} T_{s_i}(B, xB) T_{s_i}(xB, \dot{w}B) \\
&= \sharp\{xB \in G/B \mid x \in B\dot{s}_i B \cap \dot{w}B\dot{s}_i B\} \\
&= \begin{cases} \mathbf{q}, & \dot{w} = 1 \\ \mathbf{q} - 1, & \dot{w} = \dot{s}_i \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

i.e., $T_{s_i} \star T_{s_i} = (\mathbf{q} - 1) T_{s_i} + \mathbf{q}$.
Moreover, braid relations for T_{s_i} $(i = 1, 2, \ldots, d)$ hold.

Background I: Iwahori's construction of Hecke algebras

- q: an indeterminate.
- $\mathcal{A} := \mathbb{Z}[q, q^{-1}]$.
- $\mathbf{H} = \mathbf{H}(W)$: Hecke algebra over $\mathcal{A}$ generated by $H_1, H_2, \ldots, H_d$ with $(H_i - q^{-1})(H_i + q) = 0$ $(i = 1, 2, \ldots, d)$ and braid relations.
- $\widetilde{\mathbf{H}}$: (extended) affine Hecke algebra.

Theorem (Iwahori 64')

- $\mathbf{H}|_{\mathbf{q}=q^{-2}} \simeq \mathbb{Z}^{\mathbf{G}(\mathbb{F}_\mathbf{q})}[\mathscr{B} \times \mathscr{B}] \simeq \mathbb{Z}[B \backslash \mathbf{G}(\mathbb{F}_\mathbf{q})/B]$.
- $\widetilde{\mathbf{H}}|_{p=q^{-2}} \simeq \mathbb{Z}[I \backslash \mathbf{G}(\mathbb{Q}_p)/I]$, *where I is the Iwahori subgroup of $\mathbf{G}(\mathbb{Q}_p)$ over the p-adic field $\mathbb{Q}_p$.*

Background II: Equivariant K-theoretic construction

- $G = \mathbf{G}(\mathbb{C})$: complex connected reductive algebraic group.
- $\mathfrak{n} \subset \mathfrak{g} = \mathrm{Lie}(G)$: nilradical of $\mathfrak{b} = \mathrm{Lie}(B)$.
- $\mathcal{N}$: nilpotent cone of $\mathfrak{g}$, i.e. the variety of all nilpotent elements of $\mathfrak{g}$.
- $\tilde{\mathcal{N}} := T^*\mathscr{B} \simeq G \times^B \mathfrak{n}$: cotangent bundle of $\mathscr{B}$. (Springer resolution $\tilde{\mathcal{N}} \to \mathcal{N}$. It is a deep mathematical principle that the resolution at the singularity of $\mathcal{N}$ encodes rich representation-theoretic information.)
- Steinberg variety

$$\begin{aligned} Z :=& T^*\mathscr{B} \times_{\mathcal{N}} T^*\mathscr{B} \\ \simeq& \{(x, gB, g'B) \in \mathcal{N} \times \mathscr{B} \times \mathscr{B} \mid x \in (\mathrm{Ad}g)(\mathfrak{n}) \cap (\mathrm{Ad}g')(\mathfrak{n})\} \\ =& \bigsqcup_{w \in W} T^*_{\mathscr{O}_w}, \end{aligned}$$

where $T^*_{\mathscr{O}_w}$ is the conormal bundle of $\mathscr{O}_w$ in $\mathscr{B} \times \mathscr{B}$. All irreducible components of Z are $\overline{T^*_{\mathscr{O}_w}}$ $(w \in W)$.

Background II: Equivariant K-theoretic construction

- X: a G-variety.
- The 0-th equaivariant K-group
  $K^G(X) :=$ the Grothendieck group of the category of G-equivariant coherent sheaves on X.
- In particular, $K^G(pt) = R(G)$ is the representation ring of G.
- For G-varieties M_1, M_2, M_3, let $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ be G-subvarieties. Under some technical assumption, there is a convolution product

$$\star : K^G(Z_{12}) \otimes_{K^G(M_2)} K^G(Z_{23}) \to K^G(Z_{12} \circ Z_{23}),$$

where $Z_{12} \circ Z_{23}$ is the set-theoretic composition

$$Z_{12} \circ Z_{23} := \{(x_1, x_3) \in M_1 \times M_3 \mid \text{there exists } x_2 \in M_2 \text{ such that } (x_1, x_2) \in Z_{12}, (x_2, x_3) \in Z_{23}\}.$$

Background II: Equivariant K-theoretic construction

- Let $G \times \mathbb{C}\backslash\{0\}$ act on $\mathscr{B} \times \mathfrak{g}$ by $(s, z) \cdot (gB, x) = (sgB, z^2 \mathrm{Ad}_s(x))$, and then act diagonally on Z.
- $R(G \times \mathbb{C}\backslash\{0\}) \simeq R(G)[q, q^{-1}]$, where q means the tautological representation of $\mathbb{C}\backslash\{0\}$.

Theorem (Kazhdan-Lusztig 85', Ginzburg 87')

$$\widetilde{\mathbf{H}} \simeq K^{G^L \times \mathbb{C}\backslash\{0\}}(Z^L),$$

where G^L is the Langlands dual of G and Z^L is the Steinberg variety of G^L.

Langlands reciprocity:

$$\boxed{\mathbb{Z}\left[I\backslash G(\mathbb{Q}_p)/I\right] \simeq K^{G^L \times \mathbb{C}\backslash\{0\}}(Z^L)|_{p=q^{-2}}}$$

From Hecke algebras to Schur algebras

Type A:

- complete flag variety
  $\mathscr{B} = \{F = (0 = F_0 \subset F_1 \subset \cdots \subset F_d = \mathbb{F}_{\mathbf{q}}^d) \mid \dim F_i = i\}.$

$$\begin{array}{ccccc} G/B & \simeq & \text{the set of Borel subgroups} & \simeq & \mathscr{B} \\ [g] & \mapsto & g \cdot \mathfrak{b} \cdot g^{-1} & & \\ & & \mathfrak{b}_F = \{x \in \mathfrak{g} \mid x(F_i) \subset F_i, \forall i\} & \leftarrow\!\shortmid & F \end{array}$$

- *n*-step flag variety
  $\mathscr{F} = \{F = (0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = \mathbb{F}_{\mathbf{q}}^d)\} \simeq \bigsqcup_{\gamma} G/P_\gamma,$
  where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ runs over all weak composition of d into n parts, and $P_\gamma := \bigsqcup_{w \in W_\gamma} B\dot{w}B \subset G$ with
  $W_\gamma := \langle s_i \mid i \neq \gamma_1, \gamma_1 + \gamma_2, \ldots, \gamma_1 + \gamma_2 + \ldots + \gamma_{n-1} \rangle \subset W.$

What are the objects (instead of Hecke algebras) if $\mathscr{B}$ is replaced by $\mathscr{F}$?

Schur algebras of type A_{d-1}

- Schur-Jimbo duality (of type A):

$$\mathbf{U}_q(\mathfrak{gl}_n) \twoheadrightarrow \mathbf{S}_{n,d} \curvearrowright V^{\otimes d} \curvearrowleft \mathbf{H},$$

- $\mathbf{S}_{n,d} = \mathrm{End}_{\mathbf{H}}(V^{\otimes d}) = \mathrm{End}_{\mathbf{H}}(\bigoplus_\gamma x_\gamma \mathbf{H})$: the Schur algebra of type A_{d-1}, where $x_\gamma = \sum_{w \in W_\gamma} (-1)^{\ell(w)} H_w \in \mathbf{H}$ is the q-symmetrizer respect to γ.
- $\widetilde{\mathbf{S}}_{n,d} = \mathrm{End}_{\widetilde{\mathbf{H}}}(\bigoplus_\gamma x_\gamma \widetilde{\mathbf{H}})$: affine Schur algebra (of type A).

Beilinson-Lusztig-MacPherson realization

Theorem (Beilinson-Lusztig-MacPherson 90')

$$\mathbf{S}_{n,d}|_{\mathbf{q}=q^{-2}} \simeq \mathbb{Z}^{\mathbf{G}(\mathbb{F}_\mathbf{q})}[\mathscr{F} \times \mathscr{F}] \simeq \mathbb{Z}[\bigsqcup_{\gamma,\nu} P_\gamma \backslash \mathbf{G}(\mathbb{F}_\mathbf{q})/P_\nu].$$

Beilinson-Lusztig-MacPherson realization of $\mathbf{U}_q(\mathfrak{gl}_n)$: establish a stabilization property from some multiplication formulas of $\mathbf{S}_{n,d}$, by which one can construct an inverse limit algebra in which $\mathbf{U}_q(\mathfrak{gl}_n)$ embeds:

$$\begin{array}{ccc} & \mathbf{S}_{n,d} & \\ \text{stabilization} & \Downarrow & \\ & \varprojlim_{d} \mathbf{S}_{n,d} & \hookleftarrow \mathbf{U}_q(\mathfrak{gl}_n). \end{array}$$

Equivariant K-theoretic construction

- Recall the n-step flag variety $\mathscr{F} \simeq \bigsqcup_{\gamma} G/P_\gamma$, where γ runs over all composition of d into n parts. Denote $\mathscr{F}_\gamma = G/P_\gamma$.
- $\widetilde{\mathcal{N}}_{\mathbf{f}} := \bigsqcup_\gamma \widetilde{\mathcal{N}}_\gamma$ with $\widetilde{\mathcal{N}}_\gamma := T^*\mathscr{F}_\gamma \simeq G \times^{P_\gamma} \mathfrak{n}_\gamma$, where $\mathfrak{n}_\gamma$ is the nilradical of $\mathrm{Lie}(P_\gamma)$.
- Generalized Steinberg variety

$$
\begin{aligned}
Z_{\mathbf{f}} := \widetilde{\mathcal{N}}_{\mathbf{f}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}_{\mathbf{f}} &= \bigsqcup_{\gamma,\nu} \widetilde{\mathcal{N}}_\gamma \times_{\mathcal{N}} \widetilde{\mathcal{N}}_\nu \\
&= \{((gP_\gamma, x), (g'P_\nu, x)) \in \widetilde{\mathcal{N}}_\gamma \times \widetilde{\mathcal{N}}_\nu \mid \gamma, \nu \in \Lambda_{\mathbf{f}}\} \\
&\simeq \{(x, gP_\gamma, g'P_\nu) \in \mathcal{N} \times \mathscr{F}_{\mathbf{f}} \times \mathscr{F}_{\mathbf{f}} \mid \\
&\qquad \gamma, \nu \in \Lambda_{\mathbf{f}}, x \in (\mathrm{Ad} g)(\mathfrak{n}_\gamma) \cap (\mathrm{Ad} g')(\mathfrak{n}_\nu)\}.
\end{aligned}
$$

Theorem (Lusztig 99')

There is a surjective homomorphism $\mathbf{U}_q(\widetilde{\mathfrak{gl}}_n) \twoheadrightarrow \widetilde{\mathbf{S}}_{n,d}$, *with* $\widetilde{\mathbf{S}}_{n,d}|_{p=q^{-2}} \simeq \mathbb{Z}[\bigsqcup_{\gamma,\nu} I_\gamma \backslash \mathbf{G}(\mathbb{Q}_p)/I_\nu]$, *where* $I_\gamma := \bigsqcup_{w \in W_\gamma} I\dot{w}I$ *is the parahoric subgroup of* $\mathbf{G}(\mathbb{Q}_p)$ *associated with* γ.

Theorem (Ginzburg-Vasserot 93')

There is a homomorphism $\mathbf{U}_q(\widetilde{\mathfrak{gl}}_n) \to K^{G\times\mathbb{C}\backslash\{0\}}(Z_{\mathbf{f}})$.

Question:

- $K^{G\times\mathbb{C}\backslash\{0\}}(Z_{\mathbf{f}}) \simeq \widetilde{S}_{n,d}$? i.e. Langlands reciprocity?
  Yes, known to experts. (a folklore result with a folklore proof).
- What is the story other than type A?
  Note that $\mathbf{U}_q(\mathfrak{so})$ and $\mathbf{U}_q(\mathfrak{sp})$ do NOT admit a Schur-type duality with Hecke algebras.

type B/C

Theorem (Bao-Wang 18', iSchur duality)

$\mathbf{U}^\imath \curvearrowright V^{\otimes d} \curvearrowleft \mathbf{H}_{B_d}$, *where* $\mathbf{U}^\imath$ *is the coideal subalgebra of* $\mathbf{U}_q(\mathfrak{gl}_N)$ *whose classical limit is* $\mathfrak{gl}_{[N/2]} \oplus \mathfrak{gl}_{\lfloor N/2 \rfloor}$, V *is the natural module of* $\mathbf{U}_q(\mathfrak{gl}_N)$.

- $(\mathbf{U}_q(\mathfrak{gl}_N), \mathbf{U}^\imath)$ forms a quantum symmetric pair of type AIII in the sense of Lezter 99'. $\mathbf{U}^\imath$ is named an iquantum group of type AIII.
- $\mathbf{S}^\imath := \mathrm{End}_{\mathbf{H}_{B_d}}(V^{\otimes d})$, Schur algebra of type B/C.

- $\mathbf{G} = \mathbf{SO}_D$ or $\mathbf{Sp}_D$, whose natural representation space $\mathbb{F}_{\mathbf{q}}^D$ carries a nondegenerate bilinear form preserved by $\mathbf{G}$.
- $\mathscr{F}^\imath := \{F = (0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_N = \mathbb{F}_{\mathbf{q}}^D) \mid F_i = F_{N-i}^\perp\}$: the N-step isotropic flag variety.

Theorem (Bao-Kujawa-Li-Wang 18')

- $\mathbf{S}^\imath|_{\mathbf{q}=q^{-2}} \simeq \mathbb{Z}^{\mathbf{G}(\mathbb{F}_\mathbf{q})}[\mathscr{F}^\imath \times \mathscr{F}^\imath]$.
- *A Beilinson-Lusztig-MacPherson-type realization of* $\mathbf{U}^\imath$.

For example, if we take $\mathbf{G} = \mathbf{SO}_{2d+1}$ and $N = 2n+1$, then

$$\mathscr{F}^\imath \simeq \bigsqcup_{\gamma} G/P_\gamma,$$

where $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ runs over all weak compositions of d into $n+1$ parts, and $P_\gamma = \bigsqcup_{w \in W_\gamma} B\dot{w}B$ with $W_\gamma := \langle s_i \mid i \neq \gamma_0, \gamma_0 + \gamma_1, \ldots, \gamma_0 + \gamma_1 + \ldots + \gamma_{n-1} \rangle \subset W$. Here the Weyl group W is generated by $s_0, s_1, \ldots, s_{d-1}$.

affine type C

- $\widetilde{\mathbf{U}}^\imath$: iquantum group of affine type AIII. It is a coideal subalgebra of $\mathbf{U}_q(\widetilde{\mathfrak{gl}}_n)$.
- $\widetilde{\mathbf{S}}^\imath = \mathrm{End}_{\widetilde{\mathbf{H}}_{C_d}}(\widetilde{V}^{\otimes d})$: affine Schur algebra of type C, where $\widetilde{V}$ is the natural (inf. dim.) module of $\mathbf{U}_q(\widetilde{\mathfrak{gl}}_n)$.
- $\widetilde{\mathscr{F}}^\imath := \bigsqcup_\gamma \mathbf{G}(\mathbb{Q}_p)/I_\gamma$, where $I_\gamma := \bigsqcup_{w \in W_\gamma} I \dot{w} I$ is the parahoric subgroup of $\mathbf{G}(\mathbb{Q}_p)$ associated with γ.

Here is a type C counterpart of Lusztig's theorem for type A.

Theorem (Fan-Lai-Li-L-Wang 20')

- $\widetilde{\mathbf{S}}^\imath|_{p=q^{-2}} \simeq \mathbb{Z}^{\mathbf{G}(\mathbb{Q}_p)}[\widetilde{\mathscr{F}}^\imath \times \widetilde{\mathscr{F}}^\imath]$.
- *A Beilinson-Lusztig-MacPherson-type realization of* $\widetilde{\mathbf{U}}^\imath$ *via a stabilization property of* $\widetilde{\mathbf{S}}^\imath$.

- $\mathscr{F}_\gamma = G/P_\gamma$.
- $\widetilde{\mathcal{N}}_{\mathbf{f}} := \bigsqcup_\gamma \widetilde{\mathcal{N}}_\gamma$ with $\widetilde{\mathcal{N}}_\gamma := T^*\mathscr{F}_\gamma \simeq G \times^{P_\gamma} \mathfrak{n}_\gamma$.
- Generalized Steinberg variety

$$Z_{\mathbf{f}} := \widetilde{\mathcal{N}}_{\mathbf{f}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}_{\mathbf{f}} = \bigsqcup_{\gamma,\nu} \widetilde{\mathcal{N}}_\gamma \times_{\mathcal{N}} \widetilde{\mathcal{N}}_\nu.$$

Here is a type C counterpart of Ginzburg-Vasserot's theorem for type A.

Theorem (Su-Wang 24', L-Xu-Su preprint)

There is a homomorphism $\widetilde{\mathbf{U}}^\imath \to K^{G\times\mathbb{C}\backslash\{0\}}(Z_{\mathbf{f}})$.

At the Schur algebra level, the above discussion (for affine types A and C) already suggests a framework suitable for all general types.

General type

- G: reductive algebraic group, with Weyl group W.
- Hecke algebra $\mathbf{H}$ and extended affine Hecke $\widetilde{\mathbf{H}}$ of G.
- Borel subgroup $B \subset G$.
- Take any finite W-invariant subset $X_{\mathbf{f}}$ in the weight lattice.
- $\Lambda_{\mathbf{f}}$: the set of W-orbits on $X_{\mathbf{f}}$.
- For any W-orbit $\gamma \in \Lambda_{\mathbf{f}}$, let $J_\gamma = \{k | 1 \leqslant k \leqslant d, \mathbf{i}_\gamma s_k = \mathbf{i}_\gamma\}$, where $\mathbf{i}_\gamma$ is the unique anti-dominant element in γ.
- $W_\gamma = \langle s_k | k \in J_\gamma \rangle \subset W$; $P_\gamma = \bigsqcup_{w \in W_\gamma} BwB$.
- $\mathbf{T}_{\mathbf{f}} := \oplus_{\gamma \in \Lambda_{\mathbf{f}}} x_\gamma \mathbf{H}$, where $x_\gamma = \sum_{w \in W_\gamma} q^{-\ell(w)} H_w \in \mathbf{H}$.
- Schur algebra $\mathbf{S}_{\mathbf{f}} := \mathrm{End}_{\mathbf{H}}(\mathbf{T}_{\mathbf{f}})$.
- Affine Schur algebra $\widetilde{\mathbf{S}}_{\mathbf{f}} := \mathrm{End}_{\widetilde{\mathbf{H}}}(\widetilde{\mathbf{T}}_{\mathbf{f}})$ where $\widetilde{\mathbf{T}}_{\mathbf{f}} := \oplus_{\gamma \in \Lambda_{\mathbf{f}}} x_\gamma \widetilde{\mathbf{H}}$.

Example (type A_{d-1}):
Take $X_{\mathbf{f}} = \{\sum_{i=1}^d a_i \epsilon_i | a_i \in \mathbb{Z}, 1 \leqslant a_i \leqslant n\}$.
In this case, $\mathbf{T}_{\mathbf{f}} \simeq V^{\otimes d}$ and $\mathbf{S}_{\mathbf{f}} = \mathbf{S}_{n,d}$.

Realization of $\mathbf{S}_{\mathbf{f}}$ and $\widetilde{\mathbf{S}}_{\mathbf{f}}$

Theorem (L-Wang 22')

$$\mathbf{S}_{\mathbf{f}}|_{\mathbf{q}=q^{-2}} \simeq \mathbb{Z}[\bigsqcup_{\gamma,\nu\in\Lambda_{\mathbf{f}}} P_\gamma\backslash G(\mathbb{F}_{\mathbf{q}})/P_\nu].$$

Theorem (Cui-L-Wang 24')

$$\widetilde{\mathbf{S}}_{\mathbf{f}}|_{p=q^{-2}} \simeq \mathbb{Z}[\bigsqcup_{\gamma,\nu\in\Lambda_{\mathbf{f}}} I_\gamma\backslash G(\mathbb{Q}_p)/I_\nu].$$

- $\mathscr{F}_{\mathbf{f}} = \bigsqcup_{\gamma\in\Lambda_{\mathbf{f}}} \mathscr{F}_\gamma$ with $\mathscr{F}_\gamma = G/P_\gamma$.
- $\mathfrak{n}_\gamma$: the nilradical of $\mathrm{Lie}(P_\gamma)$.
- $\widetilde{\mathcal{N}}_{\mathbf{f}} := \bigsqcup_{\gamma\in\Lambda_{\mathbf{f}}} \widetilde{\mathcal{N}}_\gamma$ with $\widetilde{\mathcal{N}}_\gamma := T^*\mathscr{F}_\gamma \simeq G\times^{P_\gamma}\mathfrak{n}_\gamma$.
- $Z_{\mathbf{f}} := \widetilde{\mathcal{N}}_{\mathbf{f}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}_{\mathbf{f}} = \bigsqcup_{\gamma,\nu\in\Lambda_{\mathbf{f}}} \widetilde{\mathcal{N}}_\gamma \times_{\mathcal{N}} \widetilde{\mathcal{N}}_\nu$

Theorem (L-Xu-Yang 24')

$$\widetilde{\mathbf{S}}_{\mathbf{f}} \simeq K^{G^L\times\mathbb{C}\backslash\{0\}}(Z^L_{\mathbf{f}}).$$

Locally geometric Langlands reciprocity for affine Schur algebras

Corollary (L-Xu-Yang 24')

$$K^{G^L\times\mathbb{C}\setminus\{0\}}(Z^L_{\mathbf{f}})|_{p=q^{-2}} \simeq \mathbb{Z}[\bigsqcup_{\gamma,\nu\in\Lambda_{\mathbf{f}}} I_\gamma\backslash G(\mathbb{Q}_p)/I_\nu].$$

- Particularly, we provided a correct proof of the folklore result for affine type A (instead of the unlegal folklore proof which uses a wrong statement $x_\lambda\mathbf{H}x_\nu \simeq K^{G^L\times\mathbb{C}\setminus\{0\}}(\tilde{\mathcal{N}}_\gamma \times_{\mathcal{N}} \tilde{\mathcal{N}}_\nu)$).
- Specialize $X_{\mathbf{f}}$ a single regular W-orbit: the Langlands reciprocity for affine Hecke algebras.
- Specialize $X_{\mathbf{f}} = \{\gamma = 0\}$: the Satake isomorphism.

Unequal parameters from twisted groups

The story does not end here. We also expect unequal-parameter variants for types B, C, F and G.

Why do we study the unequal-parameter variants?

- It is not at all obvious from a geometric construction perspective, though a purely algebraic approach was provided by Lai-L 21' for finite type C and by L-Yu 25' for affine type C.
- Finite type C with two parameters provides a uniform framework for the one-parameter finite types B, C and D, while affine type C with three parameters provides a uniform framework for the one-parameter affine types B, C and D.

- $G = \mathbf{G}(\mathbb{K})$ with $\mathbb{K} = \overline{\mathbb{F}_{\mathbf{q}}}$.
- $F_{\mathbf{q}} : G \to G$: the Frobenius map relative to $\mathbb{F}_{\mathbf{q}}$.
  (For a linear group G, $F_{\mathbf{q}}((a_{ij})) = (a_{ij}^{\mathbf{q}})$).
- Take a $F_{\mathbf{q}}$-stable Borel subgroup B and an $F_{\mathbf{q}}$-stable maximal torus $T \subset B$.
- τ: a nontrivial graph automorphism of the Dynkin diagram, which induce an endomorphism of G (still denoted by τ).
- Steinberg endomorphism $F = F_{\mathbf{q}} \circ \tau : G \to G$.
  B and T are both F-stable.
- the fixed-point subgroup G^F of G, called a twisted finite group of Lie type. Similarly, B^F and T^F.

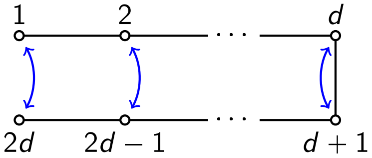

Figure 1: Type A_{2d}

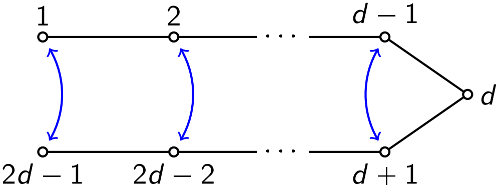

Figure 2: Type A_{2d-1}

Example: $G = GL_D(\mathbb{K})$ and $G^F = GU_D(\mathbb{F}_{q^2})$.

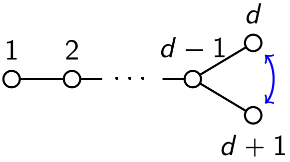

Figure 3: Type D_{d+1}

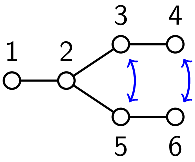

Figure 4: Type E_6

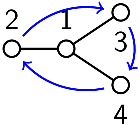

Figure 5: Type D_4

- Since T is F-stable, we can define an F-action on $W = N_G(T)/T$ by $F(gT) = F(g)T$. It is known $W^F = (N_G(T)/T)^F \simeq N_{G^F}(T)/T^F$.
- Each τ-orbit on the set of simple roots determines a parabolic subgroup of W, which has a unique longest element. Let $t_1, t_2, \ldots, t_d$ be all these longest elements.
- W^F is just the Weyl group associated with the folding Dynkin diagram, whose simple reflections are $t_1, t_2, \ldots, t_d$.

W	W^F	$t_1, t_2, \ldots, t_d$
A_{2d}	B_d	$s_1 s_{2d}, s_2 s_{2d-1}, \ldots, s_{d-1} s_{d+2}, s_d s_{d+1} s_d$
A_{2d-1}	B_d	$s_1 s_{2d-1}, s_2 s_{2d-2}, \ldots, s_{d-1} s_{d+1}, s_d$
D_{d+1}	B_d	$s_1, s_2, \ldots, s_{d-1}, s_d s_{d+1}$
E_6	F_4	$s_1, s_2, s_3 s_5, s_4 s_6$
D_4	G_2	$s_1, s_2 s_3 s_4$

- Bruhat decomposition reads

$$G^F = \bigsqcup_{w \in W^F} B^F \dot{w} B^F.$$

- $\left|B^F \dot{w} B^F\right| = \mathbf{q}^{\ell(w)} \left|B^F\right|$,
  where $\ell(w)$ is the length function of W (NOT of W^F).
- $T_{t_i} \star T_{t_i} = (\mathbf{q}^{\ell(t_i)} - 1)\, T_{t_i} + \mathbf{q}^{\ell(t_i)}$ and the braid relations still hold. This realizes the unequal-parameter Hecke algebra $\mathbb{H}$.
- $\mathbb{S}$: Schur algebra with unequal parameters.
- Similar notation W^F_γ, P^F_γ for a W^F-orbit γ in the sublattice of F-invariant weights.

Theorem (L-Song-Yang)

- $\mathbb{S}|_{\mathbf{q}=q^{-2}} \simeq \mathbb{Z}[\bigsqcup_{\gamma,\nu} P^F_\gamma \backslash G^F / P^F_\nu]$.
- *If W^F is of type B/C, we further provide a Beilinson-Lusztig-MacPherson-type realization of the iquantum group $\mathbb{U}^\imath$ of type AIII with unequal parameters.*

Equivariant K-theoretic construction with multi-parameter

The above geometric setting for twisted groups can NOT provide an equivariant K-theoretic construction for multi-parameter Schur algebras and iquantum groups. We have to employ an exotic setting introduced by Kato 09' as follows.

Affine type C with three parameters

- $G = Sp_{2d}(\mathbb{C})$ with the standard module $V = \mathbb{C}^{2d}$.
- Π: the root system with simple roots $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{d-1} = \epsilon_{d-1} - \epsilon_d, \alpha_d = 2\epsilon_d$.
- $\mathbb{V} = V^{\oplus 2} \oplus \wedge^2 V$: the exotic representation of G.
- The set of non-zero weights of $\mathbb{V}$ is

$$R = \{\pm\epsilon_i\}_{1\leqslant i\leqslant d} \cup \{\pm\epsilon_i \pm \epsilon_j\}_{1\leqslant i,j\leqslant d},$$

which corresponds one-to-one to the root system Π by a W-equivariant map

$$\Psi : \Pi \to R \quad \text{via} \quad \pm\epsilon_i \pm \epsilon_j \mapsto \pm\epsilon_i \pm \epsilon_j;\ \pm 2\epsilon_i \mapsto \pm\epsilon_i.$$

- $\mathbb{V}^+ := \bigoplus_{\alpha\in\Psi(\Pi^+)} \mathbb{V}[\alpha]$, which is a B-module.
- $\tilde{\mathcal{N}} := G \times^B \mathbb{V}^+$ (replace $\mathfrak{n}$ by $\mathbb{V}^+$ in $T^*\mathscr{B} \simeq G \times^B \mathfrak{n}$).
- $Z := \tilde{\mathcal{N}} \times_{\mathbb{V}} \tilde{\mathcal{N}}$, the exotic Steinberg variety.

Affine type C with three parameters

- Let $G \times (\mathbb{C}\backslash\{0\})^3$ acts on $\mathbb{V}$ by $(g, z_0, z_1, z_2) \cdot (v_0, v_1, v_2) \mapsto (z_0^{-1} g v_0, z_1^{-1} g v_1, z_2^{-1} g v_2)$, where $v_0, v_1 \in V$ and $v_2 \in \wedge^2 V$.
- $R(G \times (\mathbb{C}\backslash\{0\})^3) \simeq R(G)[q_0^{\pm 1}, q_1^{\pm 1}, q_2^{\pm 1}]$, where q_i $(i = 0, 1, 2)$ mean the tautological representation of $(i+1)$-th factor of $(\mathbb{C}\backslash\{0\})^3$.

Theorem (Kato 09')

The affine type C Hecke algebra with three parameters:

$$\widetilde{\mathbb{H}} \simeq K^{G \times (\mathbb{C}\backslash\{0\})^3}(Z).$$

Affine type C with three parameters

- For a weak composition $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ of d, set

$$\mathbb{V}_\lambda^+ = \bigoplus_{\alpha \in \Psi(\Pi^+) \backslash \Psi(\Pi_\lambda^+)} \mathbb{V}[\alpha],$$

  which is a P_λ-module. Here Π_λ is the root system with simple roots $\{\alpha_i \mid i \neq \lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \cdots + \lambda_n\}$.
- $\widetilde{\mathcal{N}}_{\mathbf{f}} := \bigsqcup_\lambda \widetilde{\mathcal{N}}_\lambda$ with $\widetilde{\mathcal{N}}_\lambda := G \times^{P_\lambda} \mathbb{V}_\lambda^+$ (replace $\mathfrak{n}_\lambda$ by $\mathbb{V}_\lambda^+$ in $T^*\mathscr{F}_\lambda \simeq G \times^{P_\lambda} \mathfrak{n}_\lambda$).
- The generalized exotic Steinberg variety $Z_{\mathbf{f}} := \widetilde{\mathcal{N}}_{\mathbf{f}} \times_{\mathbb{V}} \widetilde{\mathcal{N}}_{\mathbf{f}}$.

Theorem (L-Xu-Yang)

$$\widetilde{\mathbb{U}}^\imath \twoheadrightarrow \widetilde{\mathbb{S}}^\imath \simeq K^{G \times (\mathbb{C} \backslash \{0\})^3}(Z_{\mathbf{f}}).$$

Affine type BCFG with two parameters

- G: a simple algebraic group of non simply-laced type (i.e., type BCFG) defined over $\mathbf{k} = \overline{\mathbf{k}}$ with special $\mathrm{char}\ \mathbf{k}$.
- Special $\mathrm{char}\ \mathbf{k}$ means it equals the ratio's square of the lengths of the long and short roots of G, i.e.
  - $\mathrm{char}\ \mathbf{k} = 2$ when G is of type B,C or F_4;
  - $\mathrm{char}\ \mathbf{k} = 3$ when G is of type G_2.

Lemma (Hogeweij 82', Hiss 84')

If $\mathrm{char}\ \mathbf{k}$ is special, then there is a G-submodule $\mathfrak{g}_s \subset \mathfrak{g}$ whose non-zero wights are the short roots of G.

- Exotic representation: $\mathbb{V} = \mathfrak{g}_s \oplus \mathfrak{g}/\mathfrak{g}_s$, whose non-zero weights are the roots of G.

Affine type BCFG with two parameters

Using the exotic representation $\mathbb{V}$ of G, one can introduce the exotic Steinberg varieties Z and $Z_{\mathbf{f}}$ for affine type BCFG with two parameters.

Theorem (Antor 25')

The affine type BCFG Hecke algebra with two parameters:

$$\widetilde{\mathbb{H}} \simeq K^{G\times(\mathbf{k}\backslash\{0\})^2}(Z).$$

Theorem (L-Xu-Yang)

The affine type BCFG Schur algebra with two parameters:

$$\widetilde{\mathbb{S}}_{\mathbf{f}} \simeq K^{G\times(\mathbf{k}\backslash\{0\})^2}(Z_{\mathbf{f}}).$$

Thank you for your attention!