Finite-dimensional irreducible representations of quantum affine superalgebras for arbitrary parities

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Contents

Contents

Background

- Quantum groups were independently introduced by Drinfeld[1] and Jimbo[2] around 1985, commonly known as *Drinfeld-Jimbo presentation*.
- In Drinfeld-Jimbo framework, a quantum group is a q-deformation $\mathrm{U}_q(\mathfrak{a})$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{a})$ of a Kac-Moody algebra $\mathfrak{a}$.

[1] V.G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, Dokl. Akad. Nauk SSSR **283** (5) (1985) 1060–1064

[2] M. Jimbo, *A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985) 63–69

Background

- Another construction[3] of the quantized enveloping algebra $\mathrm{U}_q(\mathfrak{a})$ describes it as an associative algebra whose defining relations are expressed in terms of a R-matrix R.
- This approach, known as the *RTT presentation*, naturally equips $\mathrm{U}_q(\mathfrak{a})$ with the structure of a Hopf algebra. This presentation carries a natural comultiplication, which is useful for studying tensor products of representations.
- The matrix R here is a solution of the following quantum Yang-Baxter equation:

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12},$$

where $R^{12} := R \otimes 1$, etc.

[3] N. Reshetikhin, L. Takhtadzhyan, L. Faddeev, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. **1** (1990) 193–226

Background

- Among the major families of quantized enveloping algebras, two are especially important: Yangians and quantum affine algebras.
- In addition to the Drinfeld-Jimbo and RTT presentations, these algebras also admit a third presentation in terms of Drinfeld currents[4].
- The equivalence between Drinfeld and RTT presentations has been established for several classical Lie types. Specifically, Ding and Frenkel[5] proved it for type A, and Jing, Liu, and Molev[6,7] extended this result to type B, C, D.

[4] V.G. Drinfeld, *A new realization of Yangians and of quantum affine algebras*, Dokl. Akad. Nauk SSSR **296** (1) (1987) 13–17

[5] J. Ding, I. B. Frenkel, *Isomorphism of two realizations of quantum affine algebra*, Comm. Math. Phys. **156** (2) (1993) 277–300

[6] N. Jing, M. Liu, A. Molev, *Isomorphism between the R-matrix and Drinfeld presentations of quantum affine algebra: type C*, J. Math. Phys. **61** (3) (2020)

[7] N. Jing, M. Liu, A. Molev, *Isomorphism between the R-matrix and Drinfeld presentations of quantum affine algebra: types B and D*, SIGMA. **16** (2020) 043

Background

- Although the Drinfeld presentation does not admit a comultiplication of finite-sum type, it remains useful in representation-theoretic studies.
- Chari and Pressley[8,9] classified the finite-dimensional irreducible representations of quantum affine algebras for type A using the evaluation homomorphism

$$\mathrm{U}_q\big(\widehat{\mathfrak{sl}}_N\big) \to \mathrm{U}_q\big(\mathfrak{sl}_N\big).$$

- In addition, Gow and Molev[10] provided an alternative proof of these results using the RTT presentation.

[8] V. Chari, A. Pressley, *Quantum affine algebras*. Comm. Math. Phys. **142** (2) (1991) 261–283

[9] V. Chari, A. Pressley, *Small representations of quantum affine algebras*. Lett. Math. Phys. **30** (2) (1994) 131–145

[10] L. Gow, A. Molev, *Representations of twisted q-Yangians*. Selecta Math. (N.S.) **16** (3) (2010) 439–499

Background

- As a super symmetric generalization of quantum groups, quantum superalgebras were introduced as a powerful framework for constructing solutions to the $\mathbb{Z}_2$-graded quantum Yang-Baxter equation.
- The quantum superalgebra associated with the affine Lie superalgebra $\widehat{\mathfrak{gl}}_{m|n}$, known as the quantum affine general linear superalgebra $\mathrm{U}_q\big(\widehat{\mathfrak{gl}}_{m|n}\big)$, has been introduced via RTT presentation in several works: Fan-Hou-Shi[11], Y.-Z. Zhang[12], H.F. Zhang[13], and Jing-Li-Zhang[14] etc.

[11] H. Fan, B. Hou, K. Shi, Drinfeld constructions of the quantum affine superalgebra $U_q(\widehat{\mathfrak{gl}(m|n)})$, J. Math. Phys. **38** (1997) 411–433

[12] Y.-Z. Zhang, Comments on the Drinfeld realization of the quantum affine superalgebra $U_q[\mathfrak{gl}(m|n)^{(1)}]$ and its Hopf algebra structure, J. Phys. A **30** (1997) 8325–8335

[13] H. Zhang, RTT realization of quantum affine superalgebras and tensor products, Int. Math. Res. Not. (2016) 1126-1157

[14] N. Jing, Z. Li, J. Zhang, Quantum Berezinian for quantum affine superalgebra $U_q(\widehat{\mathfrak{gl}}_{M|N})$, Lett. Math. Phys. **115** (4) (2025) 83

Motivation

- As known, 01-sequences are used to encode the parities of generators of (affine) Lie superalgebras in $\mathfrak{gl}_{m|n}$, where 0 indicates an even index and 1 indicates an odd index.
- Unlike semisimple Lie algebras, classical Lie superalgebras contain odd roots, which means that not all Borel subalgebras are conjugate to the standard one [16].
- In fact, all the definitions of quantum affine general linear superalgebra mentioned above are based on the standard 01-sequence

$$\underbrace{00\cdots0}_{m\text{ times}}\underbrace{11\cdots1}_{n\text{ times}}.$$

[16] V.G. Kac, *Lie superalgebras*, Adv. Math. **26** (1977) 8–96

Motivation

- Nevertheless, for the supersymmetric analog of Yangian–namely, super Yangian, extensive studies involving non-standard 01-sequences are already available[17,18,19,20,21].
- While methods developed for standard 01-sequences are often not applicable to arbitrary choices of 01-sequences.
- It is natural to study finite-dimensional irreducible representations of the quantum affine general linear superalgebra $\mathrm{U}_q\big(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}}\big)$ associated with arbitrary parity sequences $\mathbf{s}$.

[17] Y.-N. Peng, *Parabolic presentations of the super Yangian $\mathrm{Y}(\mathfrak{gl}_{M|N})$ associated with arbitrary 01-sequences*, Comm. Math. Phys. **346** (2016) 313–347

[18] A. Tsymbaliuk, *Shuffle algebra realizations of type A super Yangians and quantum affine superalgebras for all Cartan data*, Lett. Math. Phys. **110** 8 (2020) 2083–2111

[19] A. Molev, *Odd reflections in the Yangian associated with $\mathfrak{gl}(m|n)$*, Lett. Math. Phys. **112** (2022) 15

[20] K. Lu, *A note on odd reflections of super Yangian and Bethe ansatz*, Lett. Math. Phys. **112** (2022) 29

[21] H. Chang, H. Hu, *A note on the center of the super Yangian $Y_{M|N}(\mathfrak{s})$*, J. Algebra **633** (2023) 648–665

Challenge and Objective

- A major challenge lies in constructing the odd reflections for $\mathrm{U}_q\big(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}}\big)$, which are essential for studying its finite-dimensional irreducible representations.
- Since this procedure is not practical for Drinfeld current generators, we use the RTT presentation—an approach inspired by the work of Gow and Molev[10,22] on quantum affine algebras and by studies on super Yangians in [19,23].
- The main goal of our work is to classify the finite-dimensional irreducible representations of $\mathrm{U}_q\big(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}}\big)$, which are isomorphic to the irreducible quotients of tensor products of evaluation representations.
- We also conjecture that every finite-dimensional irreducible representation is a tensor product of evaluation representations.

[10] L. Gow, A. Molev, *Representations of twisted q-Yangians*. Selecta Math. (N.S.) **16** (3) (2010) 439–499

[19] A. Molev, *Odd reflections in the Yangian associated with $\mathfrak{gl}(m|n)$*, Lett. Math. Phys. **112** (2022) 15

[22] A. Molev, V. N. Tolstoy, R.B. Zhang, *On irreducibility of tensor products of evaluation modules for the quantum affine algebra*, J. Phys. A Math. Gen. **37** (6) (2004) 2385

[23] R. B. Zhang, *The $gl(M|N)$ Super Yangian and Its Finite-Dimensional Representations*, Lett. Math. Phys. **37** (1996) 419–434

Contents

01-Sequences

Definition 2.1

Consider $m, n \in \mathbb{Z}_+$ with $N = m + n \geqslant 2$. We define $\mathcal{S}(m|n)$ as the set of all 01-sequences $\mathbf{s} = s_1 s_2 \cdots s_N$ that contain exactly m 0s and n 1s; any sequence $\mathbf{s} \in \mathcal{S}(m|n)$ is called a *parity sequence*. The parity sequence $\mathbf{s}$ is *standard* if

$$\mathbf{s} = \underbrace{00\cdots0}_{m \text{ times}}\underbrace{11\cdots1}_{n \text{ times}}.$$

Introduce the following two functions on the index set $I_{\mathbf{s}} = \{1, \ldots, N\}$ subject to a parity sequence $\mathbf{s}$: for $i \in I_{\mathbf{s}}$,

$$|i| = \begin{cases} \bar{0}, & \text{if } \ s_i = 0, \\ \bar{1}, & \text{otherwise.} \end{cases} \qquad d_i = \begin{cases} 1, & \text{if } \ s_i = 0, \\ -1, & \text{otherwise.} \end{cases}$$

General linear Lie superalgebra

We work over the field of complex numbers $\mathbb{C}$. Fix $\mathbf{s} \in \mathcal{S}(m|n)$, let $e_{1,\mathbf{s}}, e_{2,\mathbf{s}}, \ldots, e_{N,\mathbf{s}}$ be the standard basis of $\mathcal{V}_\mathbf{s} = \mathbb{C}^{m|n}$ with parities $|e_{i,s}| = |i|$ for all $i \in I_\mathbf{s}$. The endomorphism ring $\mathrm{End}\,\mathcal{V}_\mathbf{s}$ acts on $\mathcal{V}_\mathbf{s}$ via the rule

$$E_{ij,\mathbf{s}}(e_{k,\mathbf{s}}) = \delta_{jk} e_{i,\mathbf{s}}, \quad i, j, k \in I_\mathbf{s},$$

where $E_{ij,\mathbf{s}}$ with $|E_{ij,\mathbf{s}}| = |i| + |j|$ is the elementary matrix.

Definition 2.2

The $\mathrm{End}\,\mathcal{V}_\mathbf{s}$ forms a Lie superalgebra endowed with the super-bracket

$$[E_{ij,\mathbf{s}},\, E_{kl,\mathbf{s}}] = \delta_{jk} E_{il,\mathbf{s}} - (-1)^{(|i|+|j|)(|k|+|l|)} \delta_{il} E_{kj,\mathbf{s}}$$

for all $i, j, k, l \in I_\mathbf{s}$. In this sense, we refer to $\mathrm{End}\,\mathcal{V}_\mathbf{s}$ as the *general linear Lie superalgebra*, denoted by $\mathfrak{gl}_{m|n,\mathbf{s}}$.

Weight lattice and root lattice

- Let $\mathfrak{h}_{\mathbf{s}}$ be the span of all diagonal matrices $E_{ii,\mathbf{s}}$, and denote $\mathfrak{h}_{\mathbf{s}}$ as the *Cartan subalgebra* of $\mathfrak{g}_{\mathbf{s}}$, consider the basis $\{\varepsilon_{1,\mathbf{s}}, \dots, \varepsilon_{N,\mathbf{s}}\}$ of $\mathfrak{h}_{\mathbf{s}}^*$ such that $\varepsilon_{i,\mathbf{s}}(E_{jj,\mathbf{s}}) = \delta_{ij}$ for all $i, j \in I_{\mathbf{s}}$.
- We introduce a non-degenerate symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}_{\mathbf{s}}^*$ defined by $(\varepsilon_{i,\mathbf{s}}|\varepsilon_{j,\mathbf{s}}) = d_i\delta_{ij}$.
- For $i \in I_{\mathbf{s}} \setminus \{N\}$, we define the simple roots by $\alpha_{i,\mathbf{s}} := \varepsilon_{i,\mathbf{s}} - \varepsilon_{i+1,\mathbf{s}}$, then set $\mathbf{P}_{\mathbf{s}} := \bigoplus_{i\in I_{\mathbf{s}}} \mathbb{Z}\varepsilon_{i,\mathbf{s}}$ the *weight lattice* and $\mathbf{Q}_{\mathbf{s}} := \bigoplus_{i\in I_{\mathbf{s}}\setminus\{N\}} \mathbb{Z}\alpha_{i,\mathbf{s}}$ the *root lattice*.
- The systems of even and odd positive roots are given by

$$\Phi^+_{\bar{0},\mathbf{s}} := \{\varepsilon_{i,\mathbf{s}} - \varepsilon_{j,\mathbf{s}} \mid 1 \leqslant i < j \leqslant N \text{ and } |i| + |j| = \bar{0}\},$$
$$\Phi^+_{\bar{1},\mathbf{s}} := \{\varepsilon_{i,\mathbf{s}} - \varepsilon_{j,\mathbf{s}} \mid 1 \leqslant i < j \leqslant N \text{ and } |i| + |j| = \bar{1}\},$$

respectively.

Quantum general linear superalgebra

Let q be not a root of unity and $q_i = q^{d_i}$.

Definition 2.5 (Lin-Z 2025)

Given $\mathbf{s} \in \mathcal{S}(m|n)$, the corresponding quantum general linear superalgebra $\mathcal{U}_q(\mathfrak{gl}_{m|n,\mathbf{s}})$ (in its Drinfeld-Jimbo presentation) is an associative superalgebra. Its generators are $x^{\pm}_{i,\mathbf{s}}$ $(i \in I_{\mathbf{s}} \setminus N)$ and $k^{\pm 1}_{a,\mathbf{s}}$ $(a \in I_{\mathbf{s}})$, whose parities are defined as $|x^{\pm}_{i,\mathbf{s}}| = |i| + |i+1|$ and $|k^{\pm 1}_{a,\mathbf{s}}| = \bar{0}$. The defining relations are given as follows,

$$k_{a,\mathbf{s}} k^{-1}_{a,\mathbf{s}} = k^{-1}_{a,\mathbf{s}} k_{a,\mathbf{s}} = 1, \quad k_{a,\mathbf{s}} k_{b,\mathbf{s}} = k_{b,\mathbf{s}} k_{a,\mathbf{s}}, \tag{3.1}$$

$$k_{a,\mathbf{s}} x^{\pm}_{i,\mathbf{s}} k^{-1}_{a,\mathbf{s}} = q^{\pm(\varepsilon_{a,\mathbf{s}}|\varepsilon_{i,\mathbf{s}} - \varepsilon_{i+1,\mathbf{s}})} x^{\pm}_{i,\mathbf{s}}, \tag{3.2}$$

$$[x^{+}_{i,\mathbf{s}}, x^{-}_{i,\mathbf{s}}] = \delta_{ij} \frac{k_{i,\mathbf{s}} k^{-1}_{i+1,\mathbf{s}} - k^{-1}_{i,\mathbf{s}} k_{i+1,\mathbf{s}}}{q_i - q_i^{-1}}, \tag{3.3}$$

$$[x^{\pm}_{i,\mathbf{s}}, x^{\pm}_{j,\mathbf{s}}] = 0, \quad \text{if } (\alpha_{i,\mathbf{s}}|\alpha_{j,\mathbf{s}}) = 0, \tag{3.4}$$

$$\big[x^{\pm}_{i,\mathbf{s}}, [x^{\pm}_{i,\mathbf{s}}, x^{\pm}_{\ell,\mathbf{s}}]_{q_i}\big]_{q_i^{-1}} = 0, \quad \text{if } (\alpha_{i,\mathbf{s}}|\alpha_{i,\mathbf{s}}) \neq 0, \ \ \ell = i \pm 1, \tag{3.5}$$

$$\Big[\big[[x^{\pm}_{i-1,\mathbf{s}}, x^{\pm}_{i,\mathbf{s}}]_{q_i}, x^{\pm}_{i+1,\mathbf{s}}\big]_{q_{i+1}}, x^{\pm}_{i,\mathbf{s}}\Big] = 0, \quad \text{if } (\alpha_{i,\mathbf{s}}|\alpha_{i,\mathbf{s}}) = 0. \tag{3.6}$$

Quantum general linear superalgebra

Remark

We can characterize the *classical limit* of $\mathcal{U}_q(\mathfrak{gl}_{m|n,\mathbf{s}})$ analogously to how the standard case is treated in [24]. When $q \to 1$, $\mathcal{U}_q\big(\mathfrak{gl}_{m|n,\mathbf{s}}\big)$ coincides with the universal enveloping superalgebra $\mathcal{U}\big(\mathfrak{gl}_{m|n,\mathbf{s}}\big)$ which is obtained by the following limiting processes:

$$\lim_{q\to 1} x_{i,\mathbf{s}}^{+} = E_{i,i+1,\mathbf{s}}, \quad \lim_{q\to 1} x_{i,\mathbf{s}}^{-} = E_{i+1,i,\mathbf{s}}, \quad \lim_{q\to 1} \frac{k_{a,\mathbf{s}} - k_{a,\mathbf{s}}^{-1}}{q_a - q_a^{-1}} = E_{aa,,\mathbf{s}}.$$

[24] R. B. Zhang, *Finite-dimensional irreducible representations of the quantum supergroup* $\mathrm{U}_q(gl(m/n))$, J. Math. Phys. **34**(3) (1993) 1236–1254.

R-matrix

Definition 2.3 (Lin-Z 2025)

For $s \in \mathcal{S}(m|n)$, the (quantum) R-matrix $\widetilde{\mathcal{R}}_{q,\mathbf{s}}$ of $\mathfrak{gl}_{m|n,\mathbf{s}}$ is defined by

$$\widetilde{\mathcal{R}}_{q,\mathbf{s}} = \sum_{i,j} q_i^{-\delta_{ij}} E_{ii,\mathbf{s}} \otimes E_{jj,\mathbf{s}} - \sum_{i<j} (q_j - q_j^{-1}) E_{ij,\mathbf{s}} \otimes E_{ji,\mathbf{s}} \in \mathrm{End}\, \mathcal{V}_{\mathbf{s}}^{\otimes 2}.$$

which covers the standard case given by H.F. Zhang[13].

Lemma 2.4 (Lin-Z 2025)

The R-matrix $\widetilde{\mathcal{R}}_{q,\mathbf{s}}$ is the $\mathbb{Z}_2$-graded solution of the following quantum Yang-Baxter equation

$$\widetilde{\mathcal{R}}_{q,\mathbf{s}}^{12} \widetilde{\mathcal{R}}_{q,\mathbf{s}}^{13} \widetilde{\mathcal{R}}_{q,\mathbf{s}}^{23} = \widetilde{\mathcal{R}}_{q,\mathbf{s}}^{23} \widetilde{\mathcal{R}}_{q,\mathbf{s}}^{13} \widetilde{\mathcal{R}}_{q,\mathbf{s}}^{12}. \quad (3.7)$$

[13] H. Zhang, RTT realization of quantum affine superalgebras and tensor products, Int. Math. Res. Not. (2016) 1126-1157

Quantum general linear superalgebra

Definition 2.5 (Lin-Z 2025)

For $\mathbf{s} \in \mathcal{S}(m|n)$, the quantum general linear superalgebra $\mathrm{U}_q(\mathfrak{gl}_{m|n,\mathbf{s}})$ (in its RTT presentation) is an associative superalgebra generated by $t_{ji,\mathbf{s}}$ and $\bar{t}_{ij,\mathbf{s}}$ for $1 \leqslant i \leqslant j \leqslant N$ subject to the defining relations,

$$t_{ii,\mathbf{s}}\bar{t}_{ii,\mathbf{s}} = \bar{t}_{ii,\mathbf{s}}t_{ii,\mathbf{s}} = 1, \quad \text{for } i \in I_{\mathbf{s}}, \tag{3.8}$$

$$\mathcal{R}^{23}_{q,\mathbf{s}} T^1_{\mathbf{s}} T^2_{\mathbf{s}} = T^2_{\mathbf{s}} T^1_{\mathbf{s}} \mathcal{R}^{23}_{q,\mathbf{s}}, \tag{3.9}$$

$$\mathcal{R}^{23}_{q,\mathbf{s}} \bar{T}^1_{\mathbf{s}} \bar{T}^2_{\mathbf{s}} = \bar{T}^2_{\mathbf{s}} \bar{T}^1_{\mathbf{s}} \mathcal{R}^{23}_{q,\mathbf{s}}, \tag{3.10}$$

$$\mathcal{R}^{23}_{q,\mathbf{s}} T^1_{\mathbf{s}} \bar{T}^2_{\mathbf{s}} = \bar{T}^2_{\mathbf{s}} T^1_{\mathbf{s}} \mathcal{R}^{23}_{q,\mathbf{s}}, \tag{3.11}$$

where the matrices $T_{\mathbf{s}}$ and $\bar{T}_{\mathbf{s}}$ have the respective form

$$T_{\mathbf{s}} = \sum_{1 \leqslant i \leqslant j \leqslant N} t_{ji,\mathbf{s}} \otimes E_{ji,\mathbf{s}}, \qquad \bar{T}_{\mathbf{s}} = \sum_{1 \leqslant i \leqslant j \leqslant N} \bar{t}_{ij,\mathbf{s}} \otimes E_{ij,\mathbf{s}}.$$

The parity of generators are given by $|t_{ji,\mathbf{s}}| = |\bar{t}_{ij,\mathbf{s}}| = |i| + |j|$.

Hopf superalgebra of $U_q(\mathfrak{gl}_{m|n,\mathbf{s}})$

The superalgebra $\mathrm{U}_q(\mathfrak{gl}_{m|n,\mathbf{s}})$ possesses a Hopf superalgebra structure endowed with the comultiplication defined as

$$\triangle^{\mathbf{R}}(t_{ji,\mathbf{s}}) = \sum_{i\leqslant k\leqslant j} \varsigma_{ik;kj} t_{jk,\mathbf{s}} \otimes t_{ki,\mathbf{s}}, \quad \triangle^{\mathbf{R}}(\bar{t}_{ij,\mathbf{s}}) = \sum_{i\leqslant k\leqslant j} \varsigma_{ik;kj} \bar{t}_{ik,\mathbf{s}} \otimes \bar{t}_{kj,\mathbf{s}} \tag{3.12}$$

where $\varsigma_{ab;cd} = (-1)^{(|a|+|b|)(|c|+|d|)}$ $(a, b, c, d \in I_{\mathbf{s}})$.

Hopf superalgebra isomorphism

Proposition 2.6

The assignment

$$\bar{t}_{i,i+1,\mathbf{s}} \mapsto (q_i - q_i^{-1}) x_{i,\mathbf{s}}^{+} k_{i,\mathbf{s}},$$
$$t_{i+1,i,\mathbf{s}} \mapsto -(q_i - q_i^{-1}) k_{i,\mathbf{s}}^{-1} x_{i,\mathbf{s}}^{-},$$
$$\bar{t}_{aa,\mathbf{s}} = t_{aa,\mathbf{s}}^{-1} \mapsto k_{a,\mathbf{s}}$$

extends to a Hopf superalgebra isomorphism $\iota_{\mathbf{s}} : \mathrm{U}_q(\mathfrak{gl}_{m|n,\mathbf{s}}) \to \mathcal{U}_q(\mathfrak{gl}_{m|n,\mathbf{s}})$.

Odd reflection

Fix $\mathbf{s} \in \mathcal{S}(m|n)$ and $i \in I_{\mathbf{s}} \setminus \{N\}$. Denote $\mathbf{s}' = s'_1 \cdots s'_N := \sigma_i(\mathbf{s})$ and $d'_i = (-1)^{s'_i}$.

Proposition 2.7: Part I (Lin-Z 2025)

There exists an isomorphism $\beta_{i,\mathbf{s}} : \mathrm{U}_q\big(\mathfrak{gl}_{m|n,\mathbf{s}}\big) \to \mathrm{U}_q\big(\mathfrak{gl}_{m|n,\mathbf{s}'}\big)$ given by

$$t_{ii,\mathbf{s}} \mapsto d'_i t_{i+1,i+1,\mathbf{s}'}, \quad t_{i+1,i+1,\mathbf{s}} \mapsto d'_{i+1} t_{ii,\mathbf{s}'}, \quad t_{i+1,i,\mathbf{s}} \mapsto d'_i d'_{i+1} q^{-d'_i} \bar{t}_{i,i+1,\mathbf{s}'} \bar{t}^{-2}_{ii,\mathbf{s}'},$$

$$t_{ik,\mathbf{s}} \mapsto \varsigma'_{i-1,i;i,i+1} q^{-d'_i} t_{i+1,k,\mathbf{s}'} - \varsigma'_{k,i-1;i,i+1} t^{-1}_{ii,\mathbf{s}'} t_{i+1,i,\mathbf{s}'} t_{ik,\mathbf{s}'}, \quad \text{if } k \leqslant i-1,$$

$$t_{i+1,k,\mathbf{s}} \mapsto -\varsigma'_{i-1,i;i,i+1} d'_{i+1} t_{ik,\mathbf{s}'}, \quad \text{if } k \leqslant i-1,$$

$$t_{li,\mathbf{s}} \mapsto \varsigma'_{i,i+1;i,i+2} q^{d'_i} t_{l,i+1,\mathbf{s}'} - \varsigma'_{i,i+1;i+2,l} t_{ii,\mathbf{s}'} t_{li,\mathbf{s}'} \bar{t}_{i,i+1,\mathbf{s}'}, \quad \text{if } l \geqslant i+2,$$

$$t_{l,i+1,\mathbf{s}} \mapsto -\varsigma'_{i,i+1;i+1,i+2} d'_{i+1} t_{li,\mathbf{s}'}, \quad \text{if } l \geqslant i+2,$$

$$t_{lk,\mathbf{s}} \mapsto t_{lk,\mathbf{s}'}, \quad \text{in all remaining cases,}$$

Odd reflection

Proposition 2.7: Part II (Lin-Z 2025)

and

$$\bar{t}_{ii,\mathbf{s}} \mapsto d'_i \bar{t}_{i+1,i+1,\mathbf{s}'}, \quad \bar{t}_{i+1,i+1,\mathbf{s}} \mapsto d'_{i+1} \bar{t}_{ii,\mathbf{s}'}, \quad \bar{t}_{i,i+1,\mathbf{s}} \mapsto q^{d'_i} t^{-2}_{ii,\mathbf{s}'} t_{i+1,i,\mathbf{s}'},$$
$$\bar{t}_{ki,\mathbf{s}} \mapsto \varsigma'_{i-1,i;i,i+1} d'_i q^{d'_i} \bar{t}_{k,i+1,\mathbf{s}'} - \varsigma'_{k,i-1;i,i+1} d'_i \bar{t}_{ki,\mathbf{s}'} \bar{t}_{i,i+1,\mathbf{s}'} \bar{t}^{-1}_{ii,\mathbf{s}'}, \quad \text{if } k \leqslant i-1,$$
$$\bar{t}_{k,i+1,\mathbf{s}} \mapsto -\varsigma'_{i-1,i;i,i+1} \bar{t}_{ki,\mathbf{s}'}, \quad \text{if } k \leqslant i-1,$$
$$\bar{t}_{il,\mathbf{s}} \mapsto \varsigma'_{i,i+1;i,i+2} d'_i q^{-d'_i} \bar{t}_{i+1,l,\mathbf{s}'} - \varsigma'_{i,i+1;i+2,l} d'_i t_{i+1,i,\mathbf{s}'} \bar{t}_{il,\mathbf{s}'} \bar{t}_{ii,\mathbf{s}'}, \quad \text{if } l \geqslant i+2,$$
$$\bar{t}_{i+1,l,\mathbf{s}} \mapsto -\varsigma'_{i,i+1;i+1,i+2} \bar{t}_{il,\mathbf{s}'}, \quad \text{if } l \geqslant i+2,$$
$$\bar{t}_{kl,\mathbf{s}} \mapsto \bar{t}_{kl,\mathbf{s}'}, \quad \text{in all remaining cases},$$

where $\varsigma'_{ab;cd} = (-1)^{(|a|+|b|)(|c|+|d|)}\, (a,b,c,d \in I_{\mathbf{s}'})$.

If s contains a subsequence $s_i s_{i+1} = 00$ or 11, then $\beta_{i,\mathbf{s}}$ is an automorphism of $\mathrm{U}_q(\mathfrak{gl}_{m|n,\mathbf{s}})$; otherwise, $\beta_{i,\mathbf{s}}$ is called an *odd reflection*.

Poincaré-Birkhoff-Witt basis

Using odd reflection, we can deduce the basis of $\mathrm{U}_q(\mathfrak{gl}_{m|n,\mathbf{s}})$ from the standard case.[24]

Theorem 2.8 (Lin-Z 2025)

For any fixed $\mathrm{s} \in \mathcal{S}(m|n)$, the set of all ordered monomials

$$\overrightarrow{\prod_{i\in I_{\mathbf{s}}}} t_{i,i-1,\mathbf{s}}^{b_{i,i-1}} t_{i,i-2,\mathbf{s}}^{b_{i,i-2}} \cdots t_{i,1,\mathbf{s}}^{b_{i,1}} \times \overrightarrow{\prod_{i\in I_{\mathbf{s}}}} \bar{t}_{ii,\mathbf{s}}^{b_{ii}} \times \overrightarrow{\prod_{i\in I_{\mathbf{s}}}} \bar{t}_{1,i,\mathbf{s}}^{b_{1,i}} \bar{t}_{2,i,\mathbf{s}}^{b_{2,i}} \cdots \bar{t}_{i-1,i,\mathbf{s}}^{b_{i-1,i}}$$

with the exponents

$$b_{ij} \in \begin{cases} \mathbb{Z}_+, & \text{if } \ |i|+|j|=\bar{0} \ \text{ and } \ i\neq j, \\ \{0,1\}, & \text{if } \ |i|+|j|=\bar{1}, \\ \mathbb{Z}, & \text{if } \ i=j \end{cases}$$

forms a basis for $\mathrm{U}_q(\mathfrak{gl}_{m|n,\mathbf{s}})$.

[24] H. Lin, Y. Wang, H. Zhang, *From quantum loop superalgebras to super Yangians.* J. Algebra **650**

Contents

Highest weight representation and Kac module

To simplify the notation, we always write $\mathfrak{g}_\mathbf{s} = \mathfrak{gl}_{m|n,\mathbf{s}} (= \mathfrak{g}_\mathbf{s}(\bar{0}) \oplus \mathfrak{g}_\mathbf{s}(\bar{1}))$.

Definition 3.1

A representation V is called a *highest weight representation* over $\mathrm{U}_q(\mathfrak{g}_\mathbf{s})$ if V is generated by a non-zero vector $\zeta \in V$ such that

$$\begin{aligned} \bar{t}_{ij,\mathbf{s}}\zeta &= 0, \quad \forall\, 1 \leqslant i < j \leqslant N, \\ \bar{t}_{ii,\mathbf{s}}\zeta &= \lambda_i\zeta, \quad \lambda_i \in \mathbb{C}\backslash\{0\}. \end{aligned}$$

Set $\Lambda = (\lambda_1, \ldots, \lambda_N)$. The vector ζ and the N-tuple Λ are referred to as the *maximal vector* and the *highest weight* of V, respectively.

Let $\mathring{V}_\mathbf{s}(\Lambda)$ be the f.d. irreducible representation of $\mathrm{U}_q(\mathfrak{g}_\mathbf{s}(\bar{0}))$ with the highest weight Λ. Define the *Kac module* $K_\mathbf{s}(\Lambda)$ by setting

$$\bar{t}_{ij,\mathbf{s}}.\mathring{V}_\mathbf{s}(\Lambda) = 0.$$

The $K_\mathbf{s}(\Lambda)$ is f.d. with a unique irreducible quotient $\overline{K}_\mathbf{s}(\Lambda)$. For any given Λ, there exists a unique irreducible representation $\overline{K}_\mathbf{s}(\Lambda)$ with highest weight Λ.

Finite-dimentionality condition for standard s

Let $\mathring{V}_{\mathbf{s}}(\Lambda)$ be the finite-dimensional irreducible representation of $\mathrm{U}_q\big(\mathfrak{gl}_{m|n,\mathbf{s}}(\bar{0})\big)$ with the highest weight Λ. Define the *Kac module* $K_{\mathbf{s}}(\Lambda)$ by setting

$$\bar{t}_{ij,\mathbf{s}}.\mathring{V}_{\mathbf{s}}(\Lambda) = 0.$$

The $K_{\mathbf{s}}(\Lambda)$ is finite-dimensional with a unique irreducible quotient $\overline{K}_{\mathbf{s}}(\Lambda)$.

Let $V_{\mathbf{s}}(\Lambda)$ be a highest weight irreducible representation $\mathrm{U}_q\big(\mathfrak{gl}_{m|n,\mathbf{s}}\big)$ with highest weight Λ. If s is standard, R.B. Zhang[25] showed that $V_{\mathbf{s}}(\Lambda) \simeq \overline{K}_{\mathbf{s}}(\Lambda)$. That is to say,

Theorem 3.2

Let s be the standard parity sequence. The representation $V_{\mathbf{s}}(\Lambda)$ is finite dimensional if and only if there exist some positive integers ℓ_i $(i \neq m)$ such that

$$\frac{\epsilon_i \lambda_i}{\epsilon_{i+1}\lambda_{i+1}} = q_i^{\ell_i},$$

for some N-tuple $\epsilon = (\epsilon_1, \ldots, \epsilon_N)$ $(\forall\, \epsilon_i \in \{\pm 1\})$.

[25] R.B. Zhang, *Finite-dimensional irreducible representations of the quantum supergroup* $\mathrm{U}_q(gl(m/n))$, J. Math. Phys. **34** (3) (1993) 1236–1254

Transition rules

Let $\mathbf{s} = s_1 s_2 \cdots s_N \in \mathcal{S}(m|n)$, and let ζ be the maximal vector of $V_{\mathbf{s}}(\Lambda)$ with highest weight $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$.

Proposition 3.3 (Lin-Z 2025)

Consider $i = 1, \ldots, N-1$ such that the subsequence $s_i s_{i+1} = 01$ or 10.

(a) If the radio $\dfrac{\lambda_i}{\lambda_{i+1}} \neq \pm 1$, then the representation $V_{\mathbf{s}}(\Lambda)$ of $\mathrm{U}_q(\mathfrak{gl}_{m|n,\mathbf{s}})$ is isomorphic to the representation $V_{\sigma_i \mathbf{s}}(\Lambda')$ of $\mathrm{U}_q(\mathfrak{gl}_{m|n,\sigma_i \mathbf{s}})$, where

$$\Lambda' = (\lambda_1, \ldots, \lambda_{i-1}, q_i^{-1}\lambda_{i+1}, q_i^{-1}\lambda_i, \lambda_{i+2}, \ldots, \lambda_N).$$

(b) If the radio $\dfrac{\lambda_i}{\lambda_{i+1}} = \pm 1$, then the representation $V_{\mathbf{s}}(\Lambda)$ of $\mathrm{U}_q(\mathfrak{gl}_{m|n,\mathbf{s}})$ is isomorphic to the representation $V_{\sigma_i \mathbf{s}}(\Lambda')$ of $\mathrm{U}_q(\mathfrak{gl}_{m|n,\sigma_i \mathbf{s}})$, where

$$\Lambda' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \ldots, \lambda_N).$$

Transition rules

Based on these transition rules, we determine the finite-dimensionality of $V_{\mathbf{s}}(\Lambda)$ with $\Lambda = (\lambda_1, \ldots, \lambda_N)$ via the following recursive steps:

(1) If $\mathbf{s}$ is standard, use Theor. 3.2; otherwise, go to step (2).

(2) If there exists $1 \leqslant i < N$, $\ell < 0$ for $s_i s_{i+1} = 00$ or 11 such that

$$\frac{\lambda_i}{\lambda_{i+1}} = \pm q_i^{\ell},$$

then $V_{\mathbf{s}}(\Lambda)$ is infinite-dimensional; otherwise, go to step (3).

(3) Apply Prop. 3.3 for $s_i s_{i+1} = 01$ or 10, then set $\mathbf{s}' := \sigma_i \mathbf{s}$ and $\Lambda := \Lambda'$, and return to step (1).

Contents

Affine R-matrix

Definition 4.1 (Lin-Z 2025)

For $\mathbf{s} \in \mathcal{S}(m|n)$, the (quantum affine) R-matrix of $\mathfrak{gl}_{m|n}$ is defined by

$$\mathcal{R}_{q,\mathbf{s}}(u,v) = \mathcal{R}_{q,\mathbf{s}} u - P_{\mathbf{s}} \mathcal{R}_{q,\mathbf{s}}^{-1} P_{\mathbf{s}} v \quad \text{with} \quad P_{\mathbf{s}} = \sum_{i,j \in I_{\mathbf{s}}} (-1)^{|j|} E_{ij,\mathbf{s}} \otimes E_{ji,\mathbf{s}},$$

which covers the standard case given by H.F. Zhang[13].

Lemma 4.2 (Lin-Z 2025)

The R-matrix $\mathcal{R}_{q,\mathbf{s}}(u,v)$ is the $\mathbb{Z}_2$-graded solution of the following quantum Yang-Baxter equation

$$\mathcal{R}_{q,\mathbf{s}}^{12}(u,v)\mathcal{R}_{q,\mathbf{s}}^{13}(u,w)\mathcal{R}_{q,\mathbf{s}}^{23}(v,w) = \mathcal{R}_{q,\mathbf{s}}^{23}(v,w)\mathcal{R}_{q,\mathbf{s}}^{13}(u,w)\mathcal{R}_{q,\mathbf{s}}^{12}(u,v). \quad (5.1)$$

[13] H. Zhang, RTT realization of quantum affine superalgebras and tensor products, Int. Math. Res. Not. (2016) 1126-1157

Quantum affine general linear superalgebra

Definition 4.3 (Lin-Z 2025)

For $\mathbf{s} \in \mathcal{S}(m|n)$, the quantum affine general linear superalgebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}})$ is an associative superalgebra generated by $t_{ij,\mathbf{s}}^{(r)}$, $\bar{t}_{ij,\mathbf{s}}^{(r)}$ for $i, j \in I_\mathbf{s}$ and $r \geqslant 0$ subject to the defining relations,

$$t_{ij,\mathbf{s}}^{(0)} = \bar{t}_{ij,\mathbf{s}}^{(0)} = 0, \quad \text{if} \;\; 1 \leqslant i < j \leqslant N, \tag{5.2}$$

$$t_{ii,\mathbf{s}}^{(0)} \bar{t}_{ii,\mathbf{s}}^{(0)} = \bar{t}_{ii,\mathbf{s}}^{(0)} t_{ii,\mathbf{s}}^{(0)} = 1, \quad \text{if} \;\; i \in I_\mathbf{s}, \tag{5.3}$$

$$\mathcal{R}_{q,\mathbf{s}}^{23}(u,v) T_\mathbf{s}^1(u) T_\mathbf{s}^2(v) = T_\mathbf{s}^2(v) T_\mathbf{s}^1(u) \mathcal{R}_{q,\mathbf{s}}^{23}(u,v), \tag{5.4}$$

$$\mathcal{R}_{q,\mathbf{s}}^{23}(u,v) \bar{T}_\mathbf{s}^1(u) \bar{T}_\mathbf{s}^2(v) = \bar{T}_\mathbf{s}^2(v) \bar{T}_\mathbf{s}^1(u) \mathcal{R}_{q,\mathbf{s}}^{23}(u,v), \tag{5.5}$$

$$\mathcal{R}_{q,\mathbf{s}}^{23}(u,v) T_\mathbf{s}^1(u) \bar{T}_\mathbf{s}^2(v) = \bar{T}_\mathbf{s}^2(v) T_\mathbf{s}^1(u) \mathcal{R}_{q,\mathbf{s}}^{23}(u,v). \tag{5.6}$$

where the parities $\left|t_{ij,\mathbf{s}}^{(r)}\right| = \left|\bar{t}_{ij,\mathbf{s}}^{(r)}\right| = |i| + |j|$ and

$$T_\mathbf{s}(u) = \sum_{i,j \in I_\mathbf{s}} t_{ij,\mathbf{s}}(u) \otimes E_{ij,\mathbf{s}} \quad \text{with} \quad t_{ij,\mathbf{s}}(u) = \sum_{r \geqslant 0} t_{ij,\mathbf{s}}^{(r)} u^{-r},$$

$$\bar{T}_\mathbf{s}(u) = \sum_{i,j \in I_\mathbf{s}} \bar{t}_{ij,\mathbf{s}}(u) \otimes E_{ij,\mathbf{s}} \quad \text{with} \quad \bar{t}_{ij,\mathbf{s}}(u) = \sum_{r \geqslant 0} \bar{t}_{ij,\mathbf{s}}^{(r)} u^{r}.$$

Poincaré-Birkhoff-Witt basis

Theorem 4.4 (Lin-Z 2025)

Let $\mathcal{B}_{\mathbf{s}}$ be the set of all ordered monomials

$$\overrightarrow{\prod_{1-N\leqslant k\leqslant 1}}\ \overrightarrow{\prod_{1-k\leqslant i\leqslant N}} \left\{ \left(t^{(0)}_{i,i+k,\mathbf{s}}\right)^{b_{i,i+k,0}} \left(t^{(1)}_{i,i+k,\mathbf{s}}\right)^{b_{i,i+k,1}} \left(\bar{t}^{(1)}_{i,i+k,\mathbf{s}}\right)^{\bar{b}_{i,i+k,1}} \cdots \right\}$$

$$\times \overrightarrow{\prod_{1\leqslant i\leqslant N}} \left\{ \left(t^{(0)}_{ii,\mathbf{s}}\right)^{b_{i,i,0}} \left(\bar{t}^{(0)}_{ii,\mathbf{s}}\right)^{\bar{b}_{i,i,0}} \left(t^{(1)}_{ii,\mathbf{s}}\right)^{b_{i,i,0}} \left(\bar{t}^{(1)}_{ii,\mathbf{s}}\right)^{\bar{b}_{i,i,1}} \cdots \right\}$$

$$\times \overrightarrow{\prod_{1\leqslant k\leqslant N-1}}\ \overrightarrow{\prod_{1\leqslant i\leqslant k}} \left\{ \left(\bar{t}^{(0)}_{i,i+k,\mathbf{s}}\right)^{\bar{b}_{i,i+k,0}} \left(t^{(1)}_{i,i+k,\mathbf{s}}\right)^{b_{i,i+k,1}} \left(\bar{t}^{(1)}_{i,i+k,\mathbf{s}}\right)^{\bar{b}_{i,i+k,1}} \cdots \right\}$$

with the exponents

$$b_{i,j,r},\ \bar{b}_{i,j,r} \in \mathbb{Z}_+, \quad \text{if } |i|+|j| = \bar{0},$$
$$b_{i,j,r},\ \bar{b}_{i,j,r} \in \{0,1\}, \quad \text{if } |i|+|j| = \bar{1},$$
$$b_{i,i,0} \times \bar{b}_{i,i,0} = 0 \quad \text{for } i \in I_{\mathbf{s}}.$$

Then the monomial set $\mathcal{B}_{\mathbf{s}}$ forms an ordered basis of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}})$.

Contents

Highest weight representation

Definition 5.1

A representation V is called a *highest weight representation* over $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}})$ if V is generated by a non-zero vector $\zeta \in V$ such that

$$t_{ij,\mathbf{s}}(u)\zeta = \bar{t}_{ij,\mathbf{s}}(u)\zeta = 0, \qquad \text{for } \ 1 \leqslant i < j \leqslant N,$$
$$t_{ii,\mathbf{s}}(u)\zeta = \lambda_i(u)\zeta, \qquad \bar{t}_{ii,\mathbf{s}}(u)\zeta = \bar{\lambda}_i(u)\zeta, \qquad \text{for } \ i \in I_{\mathbf{s}},$$

where $\lambda_i(u), \bar{\lambda}_i(u)$ are the formal power series given by

$$\lambda_i(u) = \sum_{r=0}^{\infty} \lambda_i^{(r)} u^{-r}, \qquad \bar{\lambda}_i(u) = \sum_{r=0}^{\infty} \bar{\lambda}_i^{(r)} u^{r},$$

for all coefficients $\lambda_i^{(r)}, \bar{\lambda}_i^{(r)} \in \mathbb{C}$ and $\lambda_i^{(0)}\bar{\lambda}_i^{(0)} = 1$ $(i \in I_{\mathbf{s}})$. Set the N-tuples

$$\lambda(u) = (\lambda_1(u), \ldots, \lambda_N(u)), \qquad \bar{\lambda}(u) = (\bar{\lambda}_1(u), \ldots, \bar{\lambda}_N(u)).$$

The vector ζ and the pair $(\lambda(u); \bar{\lambda}(u))$ are referred to as the *maximal vector* and the *highest weights* of V, respectively.

Highest weight representation

Proposition 5.2 (Lin-Z 2025)

Every finite-dimensional irreducible representation for $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}})$ is a highest weight representation.

For any pair $(\lambda(u);\bar{\lambda}(u))$ with $\lambda_i^{(0)}\bar{\lambda}_i^{(0)}=1$ $(i\in I_{\mathbf{s}})$, there exists a non-trivial *Verma module* $M(\lambda(u);\bar{\lambda}(u))$ which is defined as a quotient of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}})$ by the left ideal generated by all coefficients of

$$\begin{aligned} &t_{ij}(u),\ \bar{t}_{ij}(u), \quad i<j,\ i,j\in I_{\mathbf{s}} \\ &t_{ii}(u)-\lambda_i(u),\ \bar{t}_{ii}(u)-\bar{\lambda}_i(u) \quad i\in I_{\mathbf{s}}. \end{aligned}$$

Then $M(\lambda(u);\bar{\lambda}(u))$ is a representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}})$ with highest weight $(\lambda(u);\bar{\lambda}(u))$. It is indecomposable and has a unique irreducible quotient $V(\lambda(u);\bar{\lambda}(u))$. Moreover, for given weights $\lambda(u)$ and $\bar{\lambda}(u)$, up to isomorphism, there is a unique highest weight irreducible representation $V(\lambda(u);\bar{\lambda}(u))$.

Evaluation homomorphism

Proposition 5.3 (Lin-Z 2025)

For any $a \in \mathbb{C} \setminus \{0\}$ and $\mathbf{s} \in \mathcal{S}(m|n)$, there exists a surjective homomorphism of superalgebras

$$\mathrm{ev}_{a,\mathbf{s}} : \mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}}) \to \mathrm{U}_q(\mathfrak{gl}_{m|n,\mathbf{s}})$$

such that

$$T_{\mathbf{s}}(u) \mapsto T_{\mathbf{s}} - \bar{T}_{\mathbf{s}} a^{-1} u^{-1}, \qquad \bar{T}_{\mathbf{s}}(u) \mapsto \bar{T}_{\mathbf{s}} - T_{\mathbf{s}} a u.$$

Proposition 5.3 is a generalization of Proposition 3.3(2) in[13] to arbitrary parity sequences. The map $\mathrm{ev}_{a,\mathbf{s}}$ serves as such an *evaluation homomorphism* for $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}})$.

[13] H. Zhang, RTT realization of quantum affine superalgebras and tensor products, Int. Math. Res. Not. (2016) 1126-1157

Evaluation representation

Let $V_{\mathbf{s}}(\mathcal{M})$ be an irreducible representation for $\mathrm{U}_q(\mathfrak{gl}_{m|n,\mathbf{s}})$ with highest weight

$$\mathcal{M} = (\mu_1, \ldots, \mu_N), \quad \text{for each } \ \mu_i \in \mathbb{C} \setminus \{0\}.$$

The *evaluation representation* $V_{a,\mathbf{s}}(\mathcal{M})$ of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}})$ is defined as the pullback of $V_{\mathbf{s}}(\mathcal{M})$ via $\mathrm{ev}_{a,\mathbf{s}}$. Consequently, the representation $V_{a,\mathbf{s}}(\mathcal{M})$ is the irreducible highest weight representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}})$ with highest weights $(\mu(u); \bar{\mu}(u))$ given by

$$\mu_i(u) = \mu_i^{-1} - \mu_i a^{-1} u^{-1}, \qquad \bar{\mu}_i(u) = \mu_i - \mu_i^{-1} a u,$$
$$\mu(u) = (\mu_1(u), \mu_2(u), \ldots, \mu_N(u)), \qquad \bar{\mu}(u) = (\bar{\mu}_1(u), \bar{\mu}_2(u), \ldots, \bar{\mu}_N(u)).$$

If $V_{\mathbf{s}}(\mathcal{M})$ is finite-dimensional, there exists a series of integers $\ell_{ij} \geqslant 0$ $(|i| + |j| = \bar{0})$ such that $\dfrac{\lambda_i}{\lambda_j} = \pm q_i^{\ell_{ij} + \#_{(i,j)}}$ by Corollary 3.3. It follows that

$$\frac{\mu_i(u)}{\mu_j(u)} = q_i^{\ell_{ij} + \#_{(i,j)}} \frac{P_{ij}(q_i^{-2} u)}{P_{ij}(u)} = \frac{\bar{\mu}_i(u)}{\bar{\mu}_j(u)},$$

where

$$P_{ij}(u) = \left(1 - \mu_j^{-2} a u\right)\left(1 - q_i^{-2} \mu_j^{-2} a u\right) \cdots \left(1 - q_i^{-2(\ell_{ij} + \#_{(i,j)} - 1)} \mu_j^{-2} a u\right).$$

Finite-dimentionality condition for standard $\mathbf{s}$

Theorem 5.4 (Lin-Z 2025)

Let $\mathbf{s}$ be the standard parity sequence. Consider the N-tuples

$$\lambda(u) = (\lambda_1(u), \lambda_2(u), \ldots, \lambda_N(u)), \quad \bar{\lambda}(u) = (\bar{\lambda}_1(u), \bar{\lambda}_2(u), \ldots, \bar{\lambda}_N(u))$$

for each series $\lambda_i(u), \bar{\lambda}_i(u)$ satisfying $\lambda_i^{(0)}\bar{\lambda}_i^{(0)} = 1$ $(i \in I_{\mathbf{s}})$. The irreducible highest weight representation $V_{\mathbf{s}}(\lambda(u); \bar{\lambda}(u))$ of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}})$ is finite-dimensional if and only if there exist a series of polynomials $P_i(u) \in 1 + u\mathbb{C}[u]$ $(i \neq m, N)$, and $Q_m(u), \widetilde{Q}_m(u)$ with the products of the constant term and the leading coefficient equal to 1, such that

$$\frac{\epsilon_i \lambda_i(u)}{\epsilon_{i+1}\lambda_{i+1}(u)} = q_i^{\deg P_i(u)} \cdot \frac{P_i(q_i^{-2}u)}{P_i(u)} = \frac{\epsilon_i \bar{\lambda}_i(u)}{\epsilon_{i+1}\bar{\lambda}_{i+1}(u)}$$

for some $\epsilon_i, \epsilon_{i+1} \in \{\pm 1\}$, and

$$\frac{\lambda_m(u)}{\lambda_{m+1}(u)} = \frac{Q_m(u)}{\widetilde{Q}_m(u)} = \frac{\bar{\lambda}_m(u)}{\bar{\lambda}_{m+1}(u)}.$$

General cases

Let $V_{\mathbf{s}}(\mathcal{M}^{(k)})$ $(k = 1, \ldots, l)$ be the finite-dimensional representation of $\mathrm{U}_q(\mathfrak{gl}_{m|n,\mathbf{s}})$ with highest weight $\mathcal{M}^{(k)} = (\mu_1^{(k)}, \ldots, \mu_N^{(k)})$. Regard the tensor product

$$V_{a_1,\mathbf{s}}(\mathcal{M}^{(1)}) \otimes V_{a_2,\mathbf{s}}(\mathcal{M}^{(2)}) \otimes V_{a_l,\mathbf{s}}(\mathcal{M}^{(l)}), \quad a_k \neq 0, \tag{6.1}$$

as a representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}})$.

Conjecture 5.5 (Lin-Z 2025)

Every finite-dimensional irreducible representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}})$ is a tensor product of evaluation representations.

General cases

Now we only checked it for the special case $(m,n)=(1,1)$.

Theorem 5.6 (Lin-Z 2025)

The representation (6.1) of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{1|1,\mathbf{s}})$ is irreducible if condition

$$\frac{a_i}{a_j} \neq \frac{\mu_{i,2}^2}{\mu_{j,1}^2} \quad \text{and} \quad \frac{\mu_{i,1}^2}{\mu_{j,2}^2} \qquad \text{for each pair } (i,j) \tag{6.2}$$

holds. Moreover, every finite-dimensional irreducible representation is isomorphic to a tensor product of evaluation representations with (6.1) satisfying (6.2).

Transition rules

Consider $i \in I_{\mathbf{s}}$ such that $s_i s_{i+1} = 01$ or 10. Let $\lambda_i(u), \lambda_{i+1}(u)$ be the formal series

$$\lambda_i(u) = \left((\lambda_i^{(1)})^{-1} - \lambda_i^{(1)} u^{-1}\right) \cdots \left((\lambda_i^{(l)})^{-1} - \lambda_i^{(l)} u^{-1}\right),$$
$$\bar{\lambda}_i(u) = \left(\lambda_i^{(1)} - (\lambda_i^{(1)})^{-1} u\right) \cdots \left(\lambda_i^{(l)} - (\lambda_i^{(l)})^{-1} u\right),$$

such that for a certain $k = 0, \ldots, l$,

$$\lambda_i^{(r)}/\lambda_{i+1}^{(s)} \neq \pm 1 \quad \text{for all} \quad r, s = 1, \ldots, k,$$
$$\lambda_i^{(r)}/\lambda_{i+1}^{(s)} \in \{\pm 1\} \quad \text{for all} \quad r = k+1, \ldots, l.$$

Proposition 5.7: Part I (Lin-Z 2025)

If the representation $V_{\mathbf{s}}(\lambda(u); \bar{\lambda}(u))$ of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n,\mathbf{s}})$ is isomorphic to the irreducible quotient of the tensor product (6.1), then $V_{\sigma_i(\mathbf{s})}(\lambda(u)'; \bar{\lambda}(u)')$ of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n,\sigma_i(\mathbf{s})})$ is finite-dimensional. Here

$$\lambda'(u) = (\lambda_1'(u), \ldots, \lambda_N'(u)), \quad \bar{\lambda}'(u) = (\bar{\lambda}_1'(u), \ldots, \bar{\lambda}_N'(u)),$$

Transition rules

Proposition 5.7: Part II (Lin-Z 2025)

with

$$\lambda'_i(u) = \Big(q_i(\lambda^{(1)}_{i+1})^{-1} - q_i^{-1}\lambda^{(1)}_{i+1}u^{-1}\Big)\cdots\Big(q_i(\lambda^{(k)}_{i+1})^{-1} - q_i^{-1}\lambda^{(k)}_{i+1}u^{-1}\Big)$$
$$\Big((\lambda^{(k+1)}_{i+1})^{-1} - \lambda^{(k+1)}_{i+1}u^{-1}\Big)\cdots\Big((\lambda^{(l)}_{i+1})^{-1} - \lambda^{(l)}_{i+1}u^{-1}\Big),$$
$$\lambda'_{i+1}(u) = \Big(q_i(\lambda^{(1)}_{i})^{-1} - q_i^{-1}\lambda^{(1)}_{i}u^{-1}\Big)\cdots\Big(q_i(\lambda^{(k)}_{i})^{-1} - q_i^{-1}\lambda^{(k)}_{i}u^{-1}\Big)$$
$$\Big((\lambda^{(k+1)}_{i})^{-1} - \lambda^{(k+1)}_{i}u^{-1}\Big)\cdots\Big((\lambda^{(l)}_{i})^{-1} - \lambda^{(l)}_{i}u^{-1}\Big),$$
$$\bar\lambda'_i(u) = \Big(q_i^{-1}\lambda^{(1)}_{i+1} - q_i(\lambda^{(1)}_{i+1})^{-1}u\Big)\cdots\Big(q_i^{-1}\lambda^{(k)}_{i+1} - q_i(\lambda^{(k)}_{i+1})^{-1}u\Big)$$
$$\Big(\lambda^{(k+1)}_{i+1} - (\lambda^{(k+1)}_{i+1})^{-1}u\Big)\cdots\Big(\lambda^{(l)}_{i+1} - (\lambda^{(l)}_{i+1})^{-1}u\Big),$$
$$\bar\lambda'_{i+1}(u) = \Big(q_i^{-1}\lambda^{(1)}_{i} - q_i(\lambda^{(1)}_{i})^{-1}u\Big)\cdots\Big(q_i^{-1}\lambda^{(k)}_{i} - q_i(\lambda^{(k)}_{i})^{-1}u\Big)$$
$$\Big(\lambda^{(k+1)}_{i} - (\lambda^{(k+1)}_{i})^{-1}u\Big)\cdots\Big(\lambda^{(l)}_{i} - (\lambda^{(l)}_{i})^{-1}u\Big),$$
$$\lambda'_p(u) = \lambda_p(u), \quad \bar\lambda'_p(u) = \bar\lambda_p(u), \quad p \neq i, i+1.$$

Transition rules

Prop. 5.7 allows us to determine the finite-dimensionality condition for $V_{\mathbf{s}}(\lambda(u);\bar{\lambda}(u))$, which is isomorphic to the irreducible quotient of a tensor product of evaluation representations.

(1) If $\mathbf{s}$ is standard, use Theor. 5.4; otherwise, go to step (2).

(2) If there do not exist some polynomial $P_i(u)$ such that one of the identities

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \pm q_i^{\deg P_i(u)} \cdot \frac{P_i(q_i^{-2}u)}{P_i(u)} = \frac{\bar{\lambda}_i(u)}{\bar{\lambda}_{i+1}(u)}$$

is not satisfied for $s_i s_{i+1} = 00$ or 11, then $V_{\mathbf{s}}(\lambda(u);\bar{\lambda}(u))$ is infinite-dimensional; otherwise, go to step (3).

(3) Apply Prop. 5.7 for $s_i s_{i+1} = 01$ or 10, then set $\mathbf{s}' := \sigma_i \mathbf{s}$ and $(\lambda(u);\bar{\lambda}(u)) := (\lambda'(u);\bar{\lambda}'(u))$, and return to step (1).

Thank for your attention!