On the finite generation of the cohomology of bosonizations

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AIM Workshop: Finite tensor categories: their cohomology and geometry; September 16-20, 2024

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(fgc2) *$H(\mathcal{C}, X) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_{\mathcal{C}}(\mathbf{1}, X)$ is finitely generated as a module over X for any object $X \in \mathcal{C}$.*

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(\Sigma X) \otimes (\Sigma Y) \arrow[r, "\cong"] \arrow[d, "\cong"'] \arrow[dr, phantom, "(-1)"] & \Sigma(X \otimes \Sigma Y) \arrow[d, "\cong"] \\
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- $\mathcal{C}$ has FGC $\Leftrightarrow$ $H(\mathcal{C}, X)$ is a noetherian module over $H(\mathcal{C}, \mathbf{1})$ for any object $X \in \mathcal{C}$.
- We say a finite-dimensional Hopf algebra has FGC if $\mathrm{rep}(K)$ has FGC. Generally, FGC can be asked for any augmented algebra or any associative algebra regarding Hochschild cohomology.

Known Results

Over a field $\Bbbk$ of $\mathrm{Char}(\Bbbk) > 0$.

- (finite groups): Golod 59', Venkov 59' and Evens 61'.
- (finite group schemes): Friedlander-Suslin 97'.
- (Drinfeld doubles of finite group schemes): Negron 21'.

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Over a noetherian commutative ring R.

- (finite group schemes$/\mathrm{Spec}(R)$): van der Kallen 23'.

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Problem (Andruskiewitsch's ABC Conjecture)

Consider an exact sequence of finite-dimensional Hopf algebras

$$\Bbbk \to A \to B \to C \to \Bbbk\,.$$

Then, A and C have FGC if and only if B has FGC.

- Exact sequences of Hopf algebras are generalizations of exact sequences of groups $1 \to K \to G \to L \to 1$.

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- For $1 \to Z(G) \to G \to G/Z(G) \to 1$ with finite p-group G, it is Evens's key step to prove $\Bbbk[G]$ has FGC.

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- Andruskiewitsch and Natale (24') showed quasi-split abelian extensions (A commutative and B cocommutative) has FGC.

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- Andruskiewitsch and Natale (24') showed quasi-split abelian extensions (A commutative and B cocommutative) has FGC.
- Known for Andruskiewitsch's ABC conjecture:

  B has FGC $\Rightarrow$ A has FGC,

  A, C have FGC $\Rightarrow$ B has FGC is widely open even for $A\#B$ (σ is trivial).

Question (Andruskiewitsch-Natale)

Let K be a finite-dimensional Hopf algebra and R a finite-dimensional braided Hopf algebra in ${}^K_K\mathcal{YD}$. If K and R have FGC, does the bosonization $R\#K$ also has FGC?

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- $H(R, \Bbbk)$ is connected graded, braided commutative in ${}^K_K\mathcal{YD}$.
- K is semisimple, $H(K\#R, \Bbbk) = H(R, \Bbbk)^K$. We only need to show the $\Bbbk$-affine algebra $H(R, \Bbbk)$ is noetherian. True: $K = \Bbbk[G]$ by Andruskiewitsch-Angiono-Pevtsova-Witherspoon (22'), $K = \Bbbk^G$ by Andruskiewitsch-Natale (25').

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- It is an open question asked by Wu-Zhang (03') that is every noetherian Hopf algebra over $\Bbbk$ an affine $\Bbbk$-algebra? True for pointed case by Yinhuo Zhang and his student Huan Jia.

Theorem (AJNOPS, 25')

The bosonization $R\#K$ has FGC if any of the following conditions are satisfied:

(1) *K is semisimple and R admits a deformation sequence.*

(2) *K is cocommutative and R admits a K-equivariant deformation sequence.*

(3) *K admits a deformation sequence $\mathfrak{C}$ of Hopf algebras and R admits a $\mathfrak{C}$-equivariant deformation sequence.*

Example

- Let G be a finite abelian group and $\mathrm{Char}(\Bbbk) > 0$. We have an exact sequence of group algebras:
  $\Bbbk \to \Bbbk[\mathbb{Z}^r] \to \Bbbk[\mathbb{Z}^r] \to \Bbbk[G] \to \Bbbk$.

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- Let $\mathfrak{g}$ be a restricted Lie algebra over $\Bbbk$ with $\mathrm{Char}(\Bbbk) = p$. We have an exact sequence of Hopf algebras $\Bbbk \to Z_0(\mathfrak{g}) \to U(\mathfrak{g}) \to u^{res}(\mathfrak{g}) \to \Bbbk$, where $u^{res}(\mathfrak{g})$ is the restricted universal enveloping algebra of $\mathfrak{g}$ and $Z_0(\mathfrak{g})$ is the p-center of $U(\mathfrak{g})$ generated by $x^p - x^{[p]}$ for any $x \in \mathfrak{g}$.

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- Lusztig's small quantum groups $u_q(\mathfrak{g})$ at q a root of unity of odd order l. We have $\Bbbk \to Z_0 \to U_q^{DK}(\mathfrak{g}) \to u_q(\mathfrak{g}) \to \Bbbk$, where $U_q^{DK}(\mathfrak{g})$ is the De Concini-Kac quantum enveloping algebra at q and Z_0 is the subalgebra generated by the l-th powers of the generators $E_\alpha, F_\alpha, K_\alpha$. Similarly for the quantum Borel $u_q(\mathfrak{b})$.

Deformation Sequences

Definition (Bezrukavnikov-Ginzburg, Negron-Pevtsova, etc.)

An augmented algebra R admits a deformation sequence if there is a pair of augmented algebra maps

$$Z \overset{j}{\hookrightarrow} Q \overset{\pi}{\twoheadrightarrow} R,$$

satisfying the following conditions:

(1) j is injective and π is surjective, preserving the augmentations.

(2) Q has finite global dimension and is module-finite and flat over Z.

(3) Z is affine central in Q and smooth at its augmentation ideal Z^+.

(4) π induces an isomorphism $R \cong Q/Z^+Q$ as augmented algebras.

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Definition (Deformation Sequences of Hopf Algebras)

A deformation sequence of Hopf algebras is an exact sequence of Hopf algebras:

$$\Bbbk \to L \to H \to K \to \Bbbk$$

satisfying

(1) H has finite global dimension and is module-finite over L.

(2) L is affine smooth central in H.

Examples of deformation sequences

Example (Drinfeld doubles of finite group schemes)

- G is a finite group scheme.
- Consider a closed embedding of G into some smooth affine group scheme H (e.g., $H = GL(V)$ for some faithful G-module V).
- $\mathcal{O}(G)$ admits a deformation sequence

$$\mathcal{O}(H/G) \hookrightarrow \mathcal{O}(H) \twoheadrightarrow \mathcal{O}(G)\,.$$

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$$\begin{tikzcd} \mathcal{O}(H/G) \arrow[r, hook] \arrow[loop above, "G"] & \mathcal{O}(H) \arrow[r, two heads] \arrow[loop above, "G"] & \mathcal{O}(G) \arrow[loop above, "G"] \end{tikzcd}$$

- There are G actions such that G acts on $\mathcal{O}(H)$ by adjoint action and on $\mathcal{O}(H/G)$ by translation.
- The bosonization $\mathcal{O}(G)\#\Bbbk[G] \cong D(G)$ is the Drinfeld double of the finite group scheme G.

Definition (Equivariant Deformation Sequences)

Suppose R is a finite-dimensional augmented algebra and K is a finite-dimensional Hopf algebra.

> We say R admits a K-equivariant deformation sequence if R admits a deformation sequence
>
> $$\begin{tikzcd} Z \arrow[loop above, "K"] \arrow[r, hook, "j"] & Q \arrow[loop above, "K"] \arrow[r, two heads, "\pi"] & R \arrow[loop above, "K"] \end{tikzcd}$$
>
> such that Z, Q and R are K-module algebras and j, π are K-module algebra maps.

Definition (Equivariant Deformation Sequences)

Suppose R is a finite-dimensional augmented algebra and K is a finite-dimensional Hopf algebra. Suppose K admits a deformation sequence of Hopf algebras

$$\mathfrak{C}: \quad W \hookrightarrow H \twoheadrightarrow K\,.$$

A deformation sequence of augmented algebras

$$Z \xhookrightarrow{j} Q \xrightarrow{\pi} R$$

is $\mathfrak{C}$-equivariant if

(1) Z, Q and R are augmented H-module algebras,

(2) j and π are maps of augmented H-module algebras,

(3) W acts trivially on Q and H acts trivially on Z.

Question&Answer Revisited

Question (Andruskiewitsch-Natale)

Let K be a finite-dimensional Hopf algebra and R a finite-dimensional braided Hopf algebra in ${}^K_K\mathcal{YD}$. If K and R have FGC, does the bosonization $R\#K$ also have FGC?

Question&Answer Revisited

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Theorem (AJNOPS, 25')

The answer to Andruskiewitsch-Natale's question is positive in any of the following conditions:

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(2) *K is cocommutative and R admits a K-equivariant deformation sequence.*

(3) *K admits a deformation sequence $\mathfrak{C}$ of Hopf algebras and R admits a $\mathfrak{C}$-equivariant deformation sequence.*

Sketch of Proof

(1) Lift the deformation sequence $Z \hookrightarrow Q \twoheadrightarrow R$ to a formal one

$$\Bbbk[[x_1, \ldots, x_n]] \cong \widehat{Z} \hookrightarrow \widehat{Q} \twoheadrightarrow \widehat{R} = R$$

by completing at the maximal ideal $\mathfrak{m} := Z^+$.

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(2) Use a result of Avramov-Gasharov-Peeva: the following are equivalent, for any finite R-modules V and W:

1. $\mathrm{Ext}^\bullet_{\widehat{Q}}(V, W)$ is finite over B_Z
2. $\mathrm{Ext}^\bullet_R(V, W)$ is finite over A_Z,

where $A_Z = S(\mathfrak{m}/\mathfrak{m}^2)$ and $B_Z = \wedge(\mathfrak{m}/\mathfrak{m}^2)$.

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where $A_Z = S(\mathfrak{m}/\mathfrak{m}^2)$ and $B_Z = \wedge(\mathfrak{m}/\mathfrak{m}^2)$.

(3) Write a DG version of Avramov-Gasharov-Peeva's result in $D^b(\mathrm{Rep}(K))$.

Corollary

Let V be a braided vector space that can be realized over a finite-dimensional Hopf algebra K. Assume that the Nichols algebra $B(V)$ is finite-dimensional and that it admits a deformation sequence

$$\mathfrak{B} : Z \hookrightarrow \tilde{B}(V) \twoheadrightarrow B(V)$$

where $\tilde{B}(V)$ is a pre-Nichols algebra of V. Then $B(V)\#K$ has FGC, provided that

- *K is semisimple; or*
- *K is cocommutative and $\mathfrak{B}$ is K-equivariant; or*
- *K admits a deformation sequence of Hopf algebras*

$$\mathfrak{C} : W \hookrightarrow H \twoheadrightarrow K$$

  such that $\mathfrak{B}$ is $\mathfrak{C}$-equivariant.

Example (The restricted Jordan plane)

- $\mathrm{Char}(\mathbb{k}) = p$ is an odd prime.
- Let (V, c) be the 2-dimensional braided vector space with a basis $\{x, y\}$ and the braiding determined by

$$\begin{aligned} c(x \otimes x) &= x \otimes x, & c(y \otimes x) &= x \otimes y, \\ c(x \otimes y) &= (y + x) \otimes x, & c(y \otimes y) &= (y + x) \otimes y. \end{aligned}$$

- The Nichols algebra $B(V)$, also called the restricted Jordan plane, is the algebra

$$B(V) = \mathbb{k}\langle x, y \mid yx - xy + \tfrac{1}{2}x^2, x^p, y^p \rangle.$$

- Suppose (V, c) can be realized in ${}^K_K\mathcal{YD}$ for some YD-triple (g, χ, η) for a finite-dimensional Hopf algebra K, where

$$\begin{aligned} &h \cdot x = \chi(h)x, \quad && h \cdot y = \chi(h)y + \eta(h)x, \quad && h \in K; \\ &\delta(x) = g \otimes x, && \delta(y) = g \otimes y. \end{aligned}$$

Example (The restricted Jordan plane)

- A YD-triple (g, χ, η) for K consists (g, χ) is a YD-pair such that

$$\chi(h)\, g = \sum \chi(h_2)\, h_1\, g\, \mathcal{S}(h_3), \qquad \text{for any } h \in K.$$

- $\eta \in \mathrm{Der}_{\chi,\chi}(K, \mathbb{k})$ such that

$$\begin{aligned} \eta(h)g &= \sum \eta(h_2)h_1 g S(h_3), \qquad \text{for any } h \in K, \\ \chi(g) &= \eta(g) = 1. \end{aligned}$$

- Then $B(V)$ admits a K-equivariant deformation sequence:

$$\mathbb{k}[x^p, y^p] \hookrightarrow \mathbb{k}\langle x, y \mid yx - xy + \tfrac{1}{2}x^2 \rangle \twoheadrightarrow B(V).$$

Example (Nichols algebras of diagonal type)

- $\mathrm{Char}(\Bbbk) = 0$ and $\theta \in \mathbb{N}$. Fix a matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta}$ whose entries are roots of 1.
- Suppose $\mathfrak{q}$ is of Cartan type, so there exists a finite Cartan matrix $\mathbf{a} = (a_{ij})_{i,j\in\mathbb{I}_\theta}$ such that $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$ for all $i \neq j \in \mathbb{I}_\theta$.
- Let V be a $\Bbbk$-vector space with a fixed basis $\{v_1, \dots, v_\theta\}$ and $V_\mathfrak{q} = (V, c)$ be the braided vector space whose braiding $c \in \mathrm{GL}(V \otimes V)$ is given by

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i \qquad \text{for any } i, j \in \mathbb{I}_\theta.$$

- There is an exact sequence of braided Hopf algebras :

$$Z^+(V_\mathfrak{q}) \overset{\iota}{\hookrightarrow} \widetilde{B}(V_\mathfrak{q}) \overset{\pi}{\twoheadrightarrow} B(V_\mathfrak{q}).$$

- $\widetilde{B}(V_\mathfrak{q})$ is the *distinguished* pre-Nichols algebra introduced by Angiono and $Z^+(V_\mathfrak{q})$ is a normal braided Hopf subalgebra of $\widetilde{B}(V_\mathfrak{q})$.

Example (Nichols algebras of diagonal type)

- Let Δ_+ be the set of positive roots corresponding to **a** with subset of simple roots $\{\alpha_i : i \in \mathbb{I}_\theta\}$.
- For $\alpha = \sum_{i\in\mathbb{I}_\theta} a_i\alpha_i$, $\beta = \sum_{i\in\mathbb{I}_\theta} b_i\alpha_i \in \Delta_+$, we set:

$$q_{\alpha\beta} = \prod_{i,j\in\mathbb{I}_\theta} q_{ij}^{a_i b_j} \qquad \text{and} \qquad N_\beta = \operatorname{ord}(q_{\beta\beta})\,.$$

- If $q_{\alpha\beta}^{N_\beta} = 1$, for all $\alpha, \beta \in \Delta_+$ ($Z^+(V_{bq})$ is central),

$$Z^+(V_\mathfrak{q}) \overset{\iota}{\hookrightarrow} \widetilde{B}(V_\mathfrak{q}) \overset{\pi}{\twoheadrightarrow} B(V_\mathfrak{q})$$

  is a deformation sequence.
- Let K be a finite-dimensional Hopf algebra that admits a family of YD-pairs $(g_i, \chi_i)_{i\in\mathbb{I}_\theta}$, $\chi_j(g_i) = q_{ij}$, for all $i, j \in \mathbb{I}_\theta$.
- The family $(g_i, \chi_i)_{i\in\mathbb{I}_\theta}$ gives a realization of $V_\mathfrak{q}$ in ${}^K_K\mathcal{YD}$. Then the above deformation sequence can be K-equivariant.

Thank You!

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