Separable algebras in higher fusion categories

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YMSC and BIMSA
joint work with Liang Kong, Zhi-Hao Zhang, Jiaheng Zhao

Algebras in monoidal 1-categories

Recall that an *algebra* or E_1*-algebra* in a monoidal 1-category $\mathcal{A}$ is a triple (A, u_A, m_A) where A is an object of $\mathcal{A}$, $u_A : 1 \to A$ and $m_A : A \otimes A \to A$ are morphisms of $\mathcal{A}$ that satisfy the unity and associtivity properties.

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An *algebra map* $f : (A, u_A, m_A) \to (B, u_B, m_B)$ is a morphism $f : A \to B$ of $\mathcal{A}$ intertwining the unit and the multiplication.

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The algebras and algebras maps in $\mathcal{A}$ form a 1-category which we denote by $\mathrm{Alg}(\mathcal{A})$ or $\mathrm{Alg}_{E_1}(\mathcal{A})$.

Commutative algebras in braided monoidal 1-categories

An *commutative algebra* or *E_2-algebra* in a braided monoidal 1-category $\mathcal{A}$ is an algebra (A, u_A, m_A) satisfying the commutative diagram

$$\begin{array}{ccccc} & & A \otimes A & & \\ & \overset{\beta_{A,A}}{\nearrow} & & \overset{m_A}{\searrow} & \\ A \otimes A & & \xrightarrow{\quad m_A \quad} & & A. \end{array}$$

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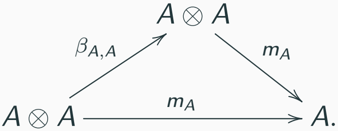

The commutative algebras and algebra maps form a 1-category which we denote by $\mathrm{CAlg}(\mathcal{A})$ or $\mathrm{Alg}_{E_2}(\mathcal{A})$. It is a full subcategory of $\mathrm{Alg}(\mathcal{A})$.

When $\mathcal{A}$ is braided, $\mathrm{Alg}(\mathcal{A})$ is monoidal under tensor product $(A, u_A, m_A) \otimes (B, u_B, m_B) = (A \otimes B, u_{A \otimes B}, m_A \otimes B)$

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\begin{array}{ccc}
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An object of $\mathrm{Alg}(\mathrm{Alg}(\mathcal{A}))$ consists of an object $(A, u_A, m_A) \in \mathrm{Alg}(\mathcal{A})$ and two morphisms $u : 1 \to A$ and $m : A \otimes A \to A$ in $\mathrm{Alg}(\mathcal{A})$.

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m is an algebra map implies that (A, u_A, m_A) is a commutative algebra. $\square$

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To summarize, we see that

- E_m-algebra $=$ E_1-algebra in the category of E_{m-1}-algebras.
- E_m-algebra $=$ E_{m+1}-algebra for large m.

Monoidal 1-categories as E_1-algebras

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- The multiplication $m_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ supplies a tensor product of $\mathcal{A}$.

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- Giving a braiding on $\mathcal{A}$ is the same thing as promoting $\otimes$ to a monoidal functor.

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An E_m-algebra $\mathcal{A}$ in $\mathcal{C}\mathrm{at}_1$ is a also symmetric monoidal 1-category for $m \geq 4$.

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E_1-algebras in an E_m-monoidal n-category $\mathcal{A}$ form an E_{m-1}-monoidal n-category $\mathrm{Alg}_{E_1}(\mathcal{A})$. Therefore, we can define by induction $\mathrm{Alg}_{E_k}(\mathcal{A}) = \mathrm{Alg}_{E_1}(\mathrm{Alg}_{E_{k-1}}(\mathcal{A}))$ for $m \geq k$.

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We denote $\mathcal{C}$ by $B\mathcal{A}$ and refer to $B\mathcal{A}$ to as the *delooping* of $\mathcal{A}$. We denote $\mathcal{A}$ by $\Omega\mathcal{C}$ and refer to $\mathcal{A}$ to as the *looping* of $\mathcal{C}$.

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Therefore, an E_m-monoidal n-category can be encoded by a sequence $(\mathcal{A}, B\mathcal{A}, B^2\mathcal{A}, \dots, B^m\mathcal{A})$ where $B^k\mathcal{A}$ is an $(n+k)$-category with a single object $\bullet$ and $B^{k-1}\mathcal{A} = \Omega(B^k\mathcal{A})$.

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The advantage of this definition is that all the algebraic structures are hidden in the categorical data.

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An *E_0-algebra map* $f : A \to B$ is a 1-morphism of $\mathcal{A}$ equipped with an equivalence $u_A \simeq u_B \circ f$.

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Definition (KZZZ)

A *separable E_k-algebra algebra* in an E_m-multi-fusion n-category $\mathcal{A}$ is a separable E_{k-1}-algebra $\hat{\Sigma}A$ in the E_{m-1}-multi-fusion $(n+1)$-category $\Sigma\mathcal{A}$ such that the unit map $1_{\mathcal{A}} \to u^R_{\hat{\Sigma}A} \circ u_{\hat{\Sigma}A}(1_{\mathcal{A}})$ extends to a condensation.

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Remark

A separable E_k-algebra in an E_m-multi-fusion n-category $\mathcal{A}$ is encoded by a sequence

$$(A, \hat{\Sigma}A, \hat{\Sigma}^2A, \ldots, \hat{\Sigma}^kA)$$

where $\hat{\Sigma}^jA$ is an E_0-algebra in $\Sigma^j\mathcal{A}$ and $\hat{\Sigma}^{j-1}A = u^R_{\hat{\Sigma}^jA} \circ u_{\hat{\Sigma}^jA}(1_{\Sigma^{j-1}\mathcal{A}})$.

E_k-modules over E_k-algebras

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- Intuitively, an E_k-algebra is an algebra with a k-dimensional multiplication, and an E_k-module receives a k-dimensional action.

Definition (KZZZ)

Let A be an E_k-algebra in an E_k-multi-fusion n-category $\mathcal{A}$. The category of *separable E_k-modules* over A is defined to be

$$\mathrm{Mod}_A^{E_k}(\mathcal{A}) := \Omega^k(\Sigma^k\mathcal{A}, \hat{\Sigma}^k A).$$

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Example

$\mathrm{Mod}_A^{E_1}(\mathcal{A}) = \Omega(\Sigma\mathcal{A}, \mathrm{RMod}_A(\mathcal{A})) \simeq \mathrm{BMod}_{A|A}(\mathcal{A})$.

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$\mathrm{Mod}_A^{E_2}(\mathcal{A}) \simeq \mathfrak{Z}_2(\mathcal{A}, \mathfrak{Z}_1(\mathrm{RMod}_A(\mathcal{A})))$. When $n = 1$, the right hand side is the category of local A-modules in $\mathcal{A}$.

Lagrangian E_k-algebras

Definition (KZZZ)

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Theorem (KZZZ)

The following conditions are equivalent for a separable E_2-algebra A in a nondegenerate braided fusion n-category $\mathcal{A}$:

1. *A is Lagrangian.*
2. $\mathrm{Mod}_A^{E_2}(\mathcal{A})$ *is trivial.*