Alterfold Theory and Modular Invariance

Zhengwei Liu
Joint with Shuang Ming, Yilong Wang, Jinsong Wu

Tsinghua University & BIMSA

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Outline

Part I: Alterfold Theory

Hopf Algebras and Tensor Categories

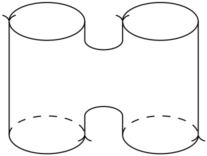
For a Hopf Algebra H, $Rep(H)$ is a tensor category.
It is called a fusion category, if it is semisimple, finite, pivotal.
Drinfeld Quantum Double: $H \rtimes \hat{H}$
$Rep(H \rtimes \hat{H})$ is the Drinfeld Center of $Rep(H)$.
The Drinfeld center of a fusion category is a modular fusion category.
It produces a representation of the modular group $PSL(2, \mathbb{Z})$.
The S-matrix is defined by the Hopf link and the T matrix is defined by the Dehn twist.

Topological Quantum Field Theory

Witten initiated Topological Quantum Field Theory (TQFT) and constructed a 2+1 TQFT using Chern-Simons theory and obtained an invariant of links in 3-manifolds as a path integral [**Wit89**], generalizing the Jones polynomial originated from subfactor theory [**Jon83**, **Jon85**, **Jon87**], and other link invariants from the representation theory of Drinfeld-Jimbo quantum groups [**Jim85**, **Dri86**, **HOMFLY85**, **PT88**, **Kau90**].

Atiyah TQFT

Atiyah 88: *Quantum* Physics and *Topology* phenomenon emerging from the continuum limit.
Atiyah's $n+1$ TQFT is defined as a symmetrical monoidal functor from **Cob** to **Vec**, which is a quantum Algebraic Topology approach to TQFT.
Object: n-manifolds without boundary $\rightarrow$ vector spaces
Morphisms: $n+1$ cobordisms $\rightarrow$ linear transformations
The TQFT is called unitary, if the partition function is reflection positive. In this case, the (finite dimensional) vector spaces are Hilbert spaces.



$$\rightarrow \quad \mathrm{Hom}(V \otimes V, V \otimes V)$$

2+1 TQFT

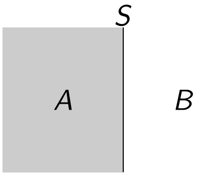
The topological invariant of Witten's 2+1 TQFT can be rigorously defined using the link invariants from quantum groups and the Lickorish-Wallace surgery theory, known as the Witten-Reshetikhin-Turaev (WRT) TQFT [**ResTur91**].

The Turaev-Viro-Barrett-Westbury (TVBW) 2+1 TQFT from a spherical fusion category [**TurVir92**, **BarWes96**] is a state sum construction over a triangulation.

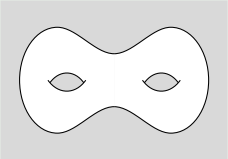
Alterfold TQFT

An n-alterfold consists of

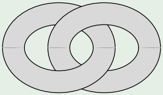
- A closed oriented n-manifold M;
- A separating hyper surface $S \subset M$;
- S separate M into A-B colored regions. (S may not be connected.)



Example



B-colored handlebody in A



A-colored Hopf Link in B

Alterfold TQFT

Definition

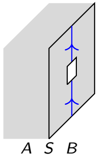
An alterfold TQFT is a symmetric monoidal functor from the cobordism category of alterfolds to the category of vector spaces.
We call A-color to be trivial, if the functor is independent of the A-color manifold.

When A-color is trivial, we focus on B-color manifolds and consider the separating surface S its space boundary. In this sense, an alterfold TQFT with trivial A-color is a TQFT with space-time boundary.
It further reduces to Atiyah's TQFT if there is no space boundary.

2+1 Alterfold TQFT

Theorem (L-Ming-Wang-Wu 23)

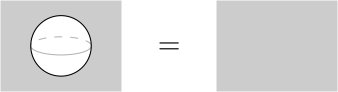
Given a spherical fusion category $\mathscr{C}$ (with Morita contexts), there is a unique 2+1 alterfold TQFT with $\mathscr{C}$-decorated separating surfaces and trivial A-color, such that the partition function of a B-color 3-disc D^3 with $\mathscr{C}$-decorated diagram Γ on the space boundary S^2 is the value of Γ in $\mathscr{C}$.



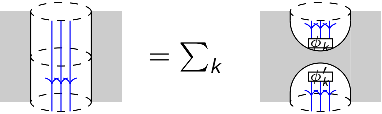
3D Skein Relations

Kirby Color: $\Omega = \sum_i d_i X_i$. Global dimension: $\mu := d(\Omega) = \sum_i d_i^2$.

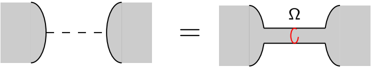
- Local Move of $\mathcal{C}$ on S
- Move 3:



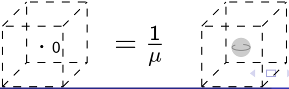
- Move 2:



- Move 1:



- Move 0:



Tube Category and Drinfeld Center

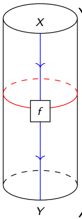
Theorem (L-Ming-Wang-Wu 23)

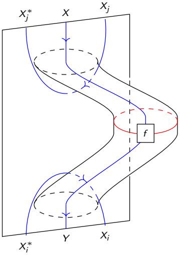
The cornered handlebodies with A-color inside and B-color outside produce a modular tensor category $\mathcal{T}$, called the tube category of $\mathcal{C}$. Moreover,

$$\mathcal{T} \overset{braided}{\simeq} \mathcal{Z}(\mathcal{C}),$$

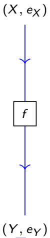
where $\mathcal{Z}(\mathcal{C})$ is the Drinfeld center of $\mathcal{C}$.

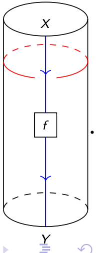
The equivalences $F : \mathcal{T} \to \mathcal{Z}(\mathcal{C})$ and $G : \mathcal{Z}(\mathcal{C}) \to \mathcal{T}$ are given as follows

$F\Bigg($


$\Bigg) = \sum_{i,j=0}^{r} d_i^{1/2} d_j^{1/2}$


,

$G\Bigg($


$\Bigg) = \frac{1}{\mu}$


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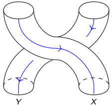
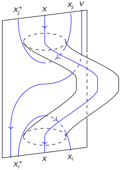
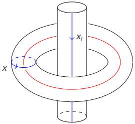
Dictionary

Drinfeld Center	Braiding	Half-Braiding	Minimal Central Idempotent
Tube Category		$\sum_{i,j} d_i^{1/2} d_j^{1/2}$ 	$\frac{d(X)}{\mu^2} \sum_{i=0}^{r}$ 

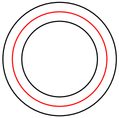
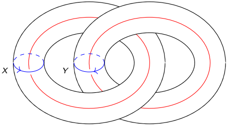
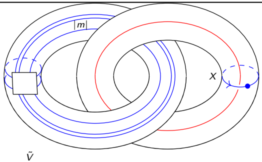
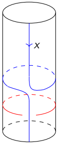
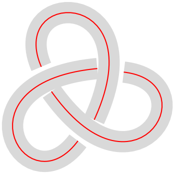
Ω-color	S-Matrix Coefficients	FS Indicators	Twist
	$\frac{1}{\mu}$ 	$\frac{1}{\mu}$ 	$\frac{1}{\mu}$ 

Figure: The Dictionary for Topologized Notions

Meanings of A/B colors

The partition function is irrelevant to the trivial A-color part, which encodes to the surgery theory in the 2+1 TQFT.



The B-color encodes the 1-dim higher center, called he bulk color.
If we run the alterfold construction twice, then the $n+2$ manifold invariants is trivial, due to the triviality of A-color. This phenomenon has been considered as the center of the center is trivial.

Theorem (L-Ming-Wang-Wu 23)

Given a spherical fusion category $\mathscr{C}$ (with Morita contexts), its alterfold TQFT contains both TVBW TQFT of $\mathscr{C}$ and WRT TQFT of the Drinfeld center of $\mathscr{C}$ in the following commuting square:

$$\begin{array}{ccc} \textit{WRT TQFT} & \subset & \textit{Alterfold TQFT} \\ \cup & & \cup \\ \textit{Atiyah TQFT} & \subset & \textit{TVBW TQFT} \end{array}$$

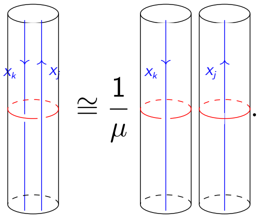
- *TVBW TQFT:*
  blowing up the skeleton of the triangulation to B-color handles.
- *WRT TQFT:*
  blowing up the framed links to A-color handles.

The equivalence of the two TQFTs on cobordisms were proved in

1. Walker 91,03 Turaev 94, Robert 95 for MTC;
2. Kawahigashi-Sato-Wakui 05 for unitary SFC;
3. Turaev-Virelizier 17, Balsam-Kirillov 10 for SFC.

When $\mathcal{C}$ is a modular fusion category, its Drinfeld center

$$\mathcal{Z}(\mathcal{C}) \cong \mathcal{C} \boxtimes \mathcal{C}^{op}.$$



Part II: Modular Invariance

Modular Invariants

In 1987, Cappelli, Itzykson and Zuber [**CIZ87**] classified minimal conformal invariant theories based on an *ADE* classification of modular invariants of quantum SU_2. The partition function Z of a torus in CFT is expended over the characters as

$$Z = \sum_{j,k} z_{jk} \chi_j \overline{\chi_k}.$$

The coefficients z_{jk} are the anomalous dimensions and they form a matrix commuting with the modular transformations S and T. The matrix $[z_{jk}]$ is called a modular invariant mass matrix.

$$ZS = SZ, \quad ZT = TZ.$$

The *ADE*-classification of modular invariants was inspired by an observation of V. Kac that the diagonal entries z_{jj} of a modular invariant are the Coxeter exponents of a Lie algebra.

ADE Classification

It is remarkable that the A_n, D_{2n}, E_6 or E_8 classification of subfactors with Jones index below 4 [**Ocn88**, **GHJ89**, **BN91**, **Izu91**, **zu94**, **Kaw95**] coincides with the *ADE* classification of modular invariants of quantum SU_2 from CFT.

- These *ADE* subfactors were classified by flat connections on the ADE dynkin diagrams, which can be interpreted as the classification of Morita contexts $\mathcal{D}$ [**Mug03**] of the unitary modular tensor category (UMTC) $\mathcal{C}$ of quantum SU_2.
- For a commutative Frobenius algebra Q in a UMTC $\mathcal{C}$, Xu [**Xu98**] introduced α-induction functors from $\mathcal{C}$ to the Morita context $\mathcal{D}$, the (Q, Q)-bimodule category over $\mathcal{C}$. Such commutative Frobenius algebras naturally arise from conformal embeddings of quantum groups.
- Böckenhauer, Evans and Kawahigashi systematically studied the modular invariance of a commutative Frobenius algebras Q in UMTC based on α-inductions in [**BE98**, **BE99**, **BE00**, **BE99b**, **BEK99**, **BEK00**].

BEK theory

Theoretical results of Böckenhauer, Evans and Kawahigashi on modular invariants for a commutative Frobenius algebra in UMTC are summarized in [**BEK00**] as:

1. Surjectivity of the α-induction in $\mathcal{D}$;
2. The Z-matrix Z is a modular invariance;
3. Characterize the matrix units of dual fusion algebra;
4. The Grothendieck ring $K_0(\mathcal{D})$ is commutative if and only if $z_{jk} \in \{0, 1\}$;
5. The number of simple objects in $\mathcal{D}$ is $Tr(ZZ^t)$;
6. The diagonal entry Z_{jj} is the dimension of the j^{th} eigenspace of NIMrep, i.e., $K_0(\mathcal{C})$ acting on $K_0(\mathcal{M})$, where $\mathcal{M}$ is the category of $\mathcal{C}$-$\mathcal{D}$ bimodules;
7. The number of simple objects in $\mathcal{M}$ is $Tr(Z)$.

Main Theorems

In joint work [**L-Ming-Wang-Wu 24**] arxiv:2412.12702, we propose a topological partition function and its modular invariant theory based on the alterfold theory of a modular fusion category $\mathcal{C}$.

We provide streamlined quick proofs and broad generalizations of above theoretical results Böckenhauer, Evans and Kawahigashi summarized in [**BEK00**].

We generalize a recent result of Kawahigashi [**Kaw23**, **Kaw24**] that a Frobenius algebra in a UMTC is commutative iff its induced connection is flat.

Our results work for spherical Morita contexts of modular fusion categories over a general field, without the unitary assumption.

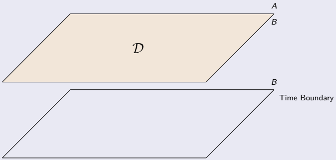
Additionally, we introduce the concept of double α-induction for pairs of Morita contexts and define its higher-genus Z-transformation, which remains invariant under the action of the mapping class group.

We also establish a novel integral identity for modular invariance across multiple Morita contexts, unifying several known identities as special cases.

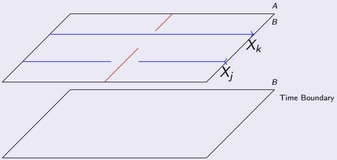
Topological Partition Function

Definition

For a modular fusion category $\mathcal{C}$ and its spherical Morita context $\mathcal{D}$, we construct the partition function $Z^{\mathcal{D}}$ in alterfold TQFT and define its modular invariance $[z_{jk}]$ as



$$= \sum_{j,k} z_{jk} \frac{1}{\mu}$$



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where μ is the global dimension of $\mathcal{C}$.

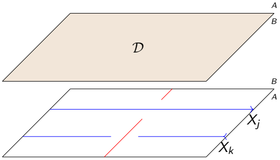
It is a topological analogue of the CFT partition function

$$Z = \sum_{j,k} z_{jk} \chi_j \overline{\chi_k}.$$

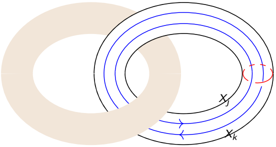
Topological Modular Invariance

The modular invariance property of the matrix $Z^{\mathcal{D}}$ is transparent from its topological nature. The integer entry z_{jk} is

$$z_{jk} = \frac{1}{\mu^2}$$



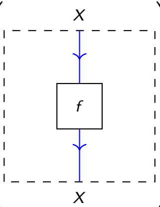
$$= \frac{1}{\mu^2}$$

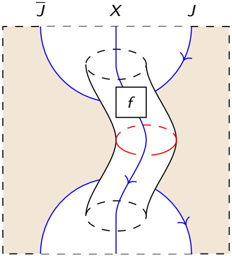


$$= \dim \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(I(\mathbf{1}_{\mathcal{D}})), X_j \boxtimes X_k^{op}),$$

where I is the induction functor.

The α-induction functors $\alpha_\pm$ in the alterfold TQFT:

α_+ (


) $= \frac{1}{d_J}$


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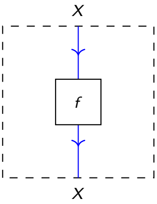
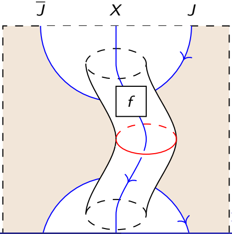
α_- (


) $= \frac{1}{d_J}$


. (1)

BEK Z-matrix

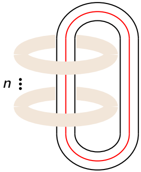
Through the α-induction functors, we have that

$$
\begin{aligned}
z^{\mathcal{D}}_{jk} &= \dim \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(I(\mathbf{1}_{\mathcal{D}}), X_j \boxtimes X^{op}_k) \\
&= \dim \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}_{\mathcal{D}}), F(X_j \boxtimes X^{op}_k)) \\
&= \dim \mathrm{Hom}_{\mathcal{D}}(\alpha_+(X_j), \alpha_-(X_k)).
\end{aligned}
$$

The equality $z^{\mathcal{D}}_{jk} := \dim \mathrm{Hom}_{\mathcal{D}}(\alpha_+(X_j), \alpha_-(X_k))$ was taken as the definition of the modular invariant Z-matrix in the theory of Böckenhauer-Evans-Kawahigashi. The modular invariant property is not transparent from that definition.

We have the following identities for modular invariants from Morita contexts, which can be applied as obstructions in the classification of Morita contexts of a modular fusion category. For example,

$$\sum_{i,j}(z_{ij}^{\mathcal{D}})^n\left(\frac{\mu}{d_i d_j}\right)^{n-2} = \frac{1}{\mu^2}$$



$$= \dim \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(1_{\mathcal{Z}(\mathcal{C})}, I(1_{\mathcal{D}})^{\otimes n})$$

Integrality

Theorem (L-Ming-Wang-Wu 24)

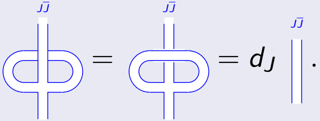
1. $$\sum_{j,k=1}^{r}\prod_{s=1}^{n} z_{jk}^{\mathcal{D}_s}\frac{\mu^{n-2}}{d_k^{n-2}d_j^{n-2}} = \dim \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathbf{1}_{\mathcal{Z}(\mathcal{C})}, I(\mathbf{1}_{\mathcal{D}_1})\otimes\cdots\otimes I(\mathbf{1}_{\mathcal{D}_n})).$$
2. $$\sum_{j=1}^{r}\prod_{s=1}^{n} z_{jj}^{\mathcal{D}_s}\frac{\mu^{n-2}}{d_j^{2n-4}} = \dim \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(I(\mathbf{1}_{\mathcal{C}}), I(\mathbf{1}_{\mathcal{D}_1})\otimes\cdots\otimes I(\mathbf{1}_{\mathcal{D}_n})).$$
3. $$\sum_{j=1}^{r}\prod_{s=1}^{n} z_{j1}^{\mathcal{D}_s}\frac{\mu^{n-2}}{d_j^{n-2}} = \dim \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(\sum_{j} G^{+}(X_j), I(\mathbf{1}_{\mathcal{D}_1})\otimes\cdots\otimes I(\mathbf{1}_{\mathcal{D}_n})).$$
4. $$\sum_{j=1}^{r}\prod_{s=1}^{n} z_{1j}^{\mathcal{D}_s}\frac{\mu^{n-2}}{d_j^{n-2}} = \dim \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(\sum_{j} G^{-}(X_j), I(\mathbf{1}_{\mathcal{D}_1})\otimes\cdots\otimes I(\mathbf{1}_{\mathcal{D}_n})).$$

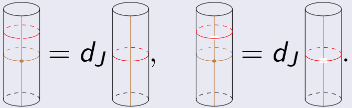
Example: $\sum_{j,k=1}^{r} z_{jk}^{\mathcal{D}} z_{jk}^{\mathcal{E}}$ is the number of irreducible $\mathcal{D}-\mathcal{E}$ bimodules.

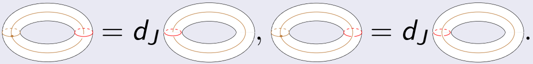
Theorem (L-Ming-Wang-Wu 24)

Suppose that the braided fusion category $\mathcal{C}$ is unitary and $Q = J\bar{J}$ is a Frobenius algebra in $\mathcal{C}$. Then the following statements are equivalent:

(1) *Q is commutative,*

(2) 

(3) 

(4) 

(5) *the α-induced bi-unitary connection is flat.*

(1) $\iff$ (5) is the main theorem of Kawahigashi [**Kaw23**, **Kaw24**].
Our proof of (1) $\iff$ (5) does not require the unitary condition.

Summary and Outlook

The alterfold 2+1 TQFT provides a natural and unified framework to study various concepts in (unitary/modular) fusion categories, including Morita contexts, Drinfeld center, full center, flat connections, Frobenius algebras, α-inductions, modular invariants, Lagrangian algebras, WRT TQFT, TVBW TQFT etc.

The modular invariant theory is a cornerstone in 1+1 CFT. It will be interesting to understand other topological properties of 1+1 CFT using the alterfold TQFT.

Thank You!