TENSOR NETWORK STATES
AND SPHERICAL BICATEGORIES

Hopf Algebras and Tensor Categories

22 January 2026

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Plan

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☞ **motivation : gapped ground states**

of quantum many-body systems

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- ☞ **motivation :** **gapped ground states**
- ☞ **physics :** **1-d systems :** **MPS**
  - **2-d systems :** **PEPS**
  - **symmetries :** **MPO**

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- ☞ **(bi)categorical perspective**

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- ☞ **state-sum models with defects**

based on

2008.11187 – with L. Lootens, J. Haegeman, C. Schweigert, F. Verstraete
2207.07031 – with C. Galindo, D. Jaklitsch, C. Schweigert
ongoing – with Y. Ogata, C. Schweigert

Motivation

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quantum many-body system :

- collection of *sites* with adjancency rules (*lattice* of atoms/molecules)
- at each site a *state space* $\mathcal{H}$:

  finite-dimensional vector space with non-degenerate pairing $\mathcal{H} \otimes \mathcal{H} \longrightarrow \mathbb{k}$
- total state space $\mathcal{H}_{\rm tot} = \mathcal{H}^{\otimes N}$ with $N \gg 1$
- dynamics / interactions specified by a *Hamilton operator* $H : \mathcal{H}_{\rm tot} \longrightarrow \mathcal{H}_{\rm tot}$

  e.g. nearest-neighbour Heisenberg Hamiltonian

  (largely immaterial in the sequel)

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result :

☞ methods for parametrizing states in a small subspace of $\mathcal{H}_{\text{tot}}$ which e.g. give excellent approximation to the ground state

1-d: Spin chains

Spin chains

1-d system : **spin chain**

- collection of sites : along a line
- for convenience : line $\rightsquigarrow$ circle (“periodic boundary conditions”)

  and translationally invariant Hamilton operator

Spin chains

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tool :

- auxiliary vector space $\mathcal{V}$
- $D \times D \times d$-tensor : numbers $\boxed{(A^j)_{p,q}}$ with $j \in \{1, 2, \ldots, h = \dim(\mathcal{H})\}$ and $p, q \in \{1, 2, \ldots, D = \dim(\mathcal{V})\}$
- family of states

$$\boxed{|\psi(A)\rangle = \sum_{j_1, j_2, \ldots, j_N = 1}^{h} \mathrm{Tr}(A^{j_1} A^{j_2} \cdots A^{j_N})\, |j_1\rangle \otimes |j_2\rangle \cdots \otimes |j_N\rangle \in \mathcal{H}_{\mathrm{tot}}}$$

with $\{\, |j\rangle\}$ a basis of $\mathcal{H}$

depending on $D^2 h \ll h^N = \dim(\mathcal{H}_{\mathrm{tot}})$ parameters

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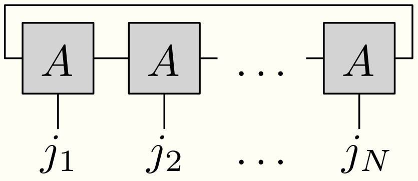
graphically :

$|\psi(A)\rangle \;=$



Spin chains

terminology : **MPS** $\equiv$ matrix product state

result :

- MPS give efficient approximation to ground states of local gapped Hamiltonians
- MPS can be easily studied numerically

challenge :

get a *conceptual* handle on the subspace spanned by the MPS vectors $|\psi(A)\rangle$

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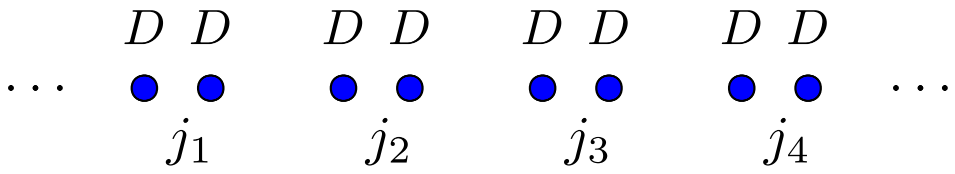
alternative terminology : originating from alternative construction

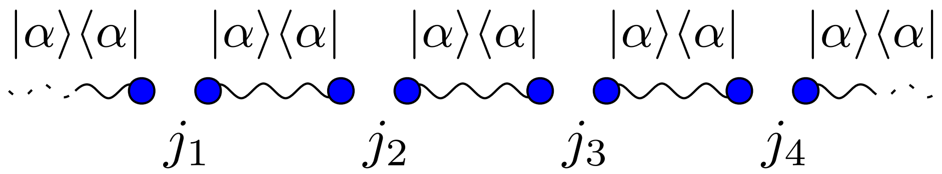
Interlude: MPS $=$ PEPS

alternative construction :

- at each site place two ***D***- dim degrees of freedom :



- *maximally entangle* all pairs on neighboring sites :



with $|\alpha\rangle = \sum_{m=1}^{D} |m\rangle \otimes |m\rangle$

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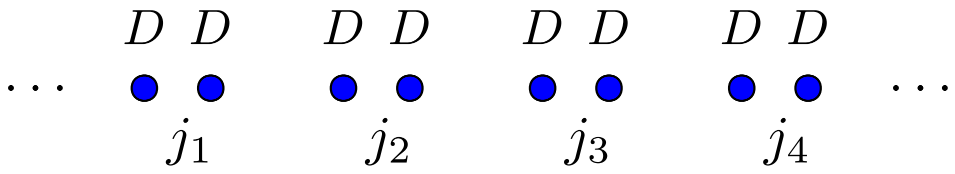

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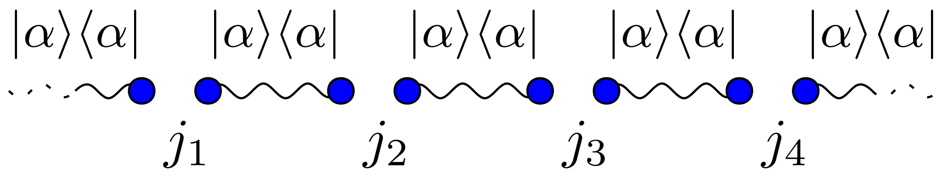
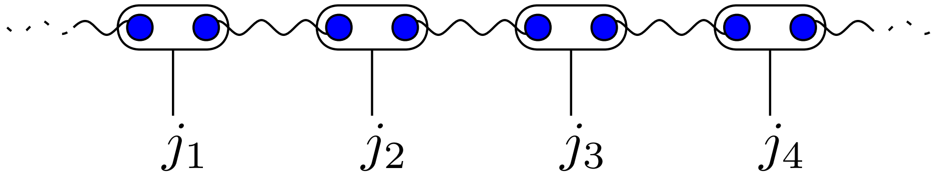


- act on the pair at each site with the linear map $f_A : \mathbb{C}^D \otimes \mathbb{C}^D \to \mathbb{C}^h$



$\Longrightarrow$ realize the vector $|\psi(A)\rangle$ as projected entangled pair state

Spin chains

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important virtue of the description as PEPS : generalizes directly to $d > 1$

2-d: PEPS

PEPS

2-d system :

- collection of sites : on a plane
- 2-d adjancency rules : each site with n nearest neighbors
- at each site physical state space $\mathcal{H}$ & n copies of auxiliary vector space $\mathcal{V}$

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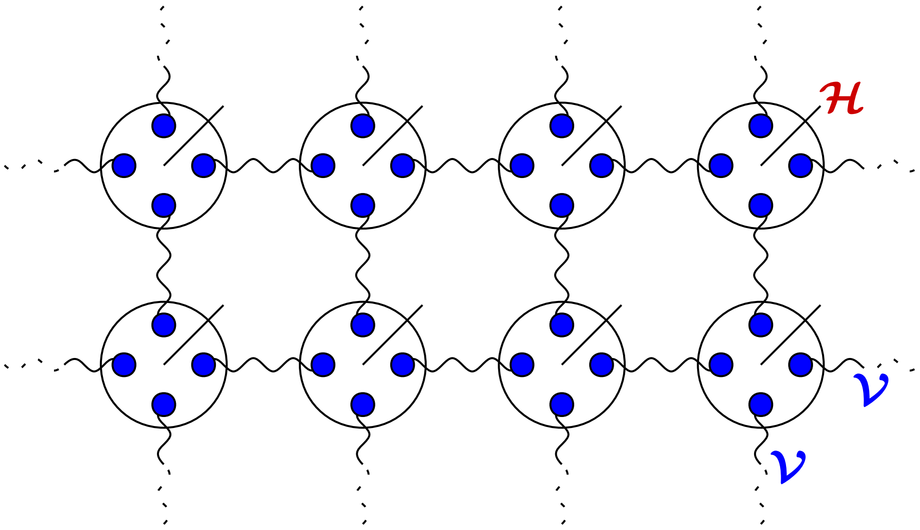
e.g. for square lattice :



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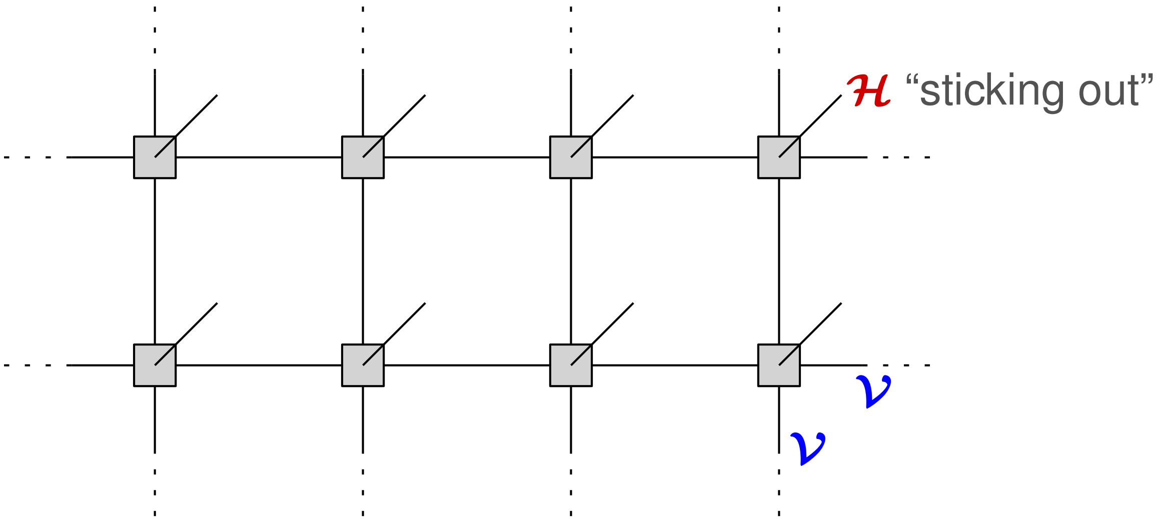

Line defects

idea : more structure via topological symmetries

- explaining e.g. the topology-dependence of ground-state degeneracies
- naturally encoded in codimension - 1 defects

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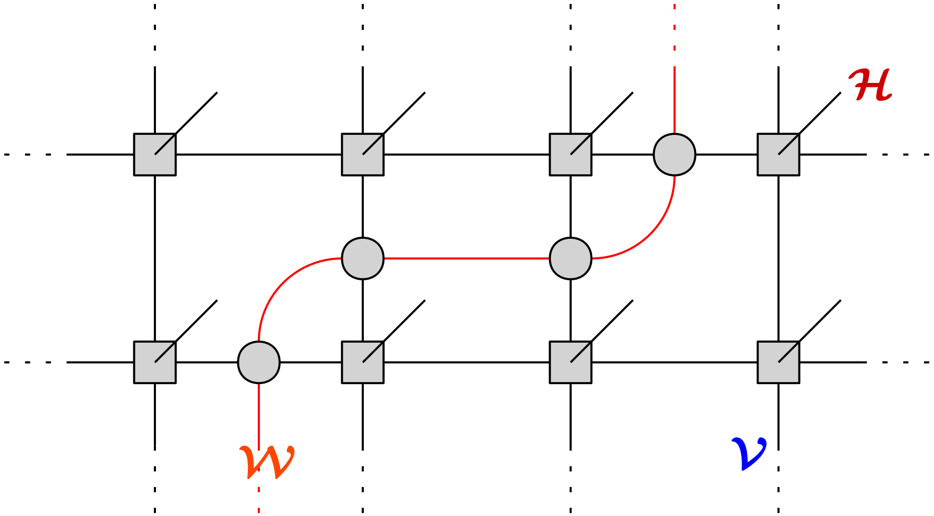

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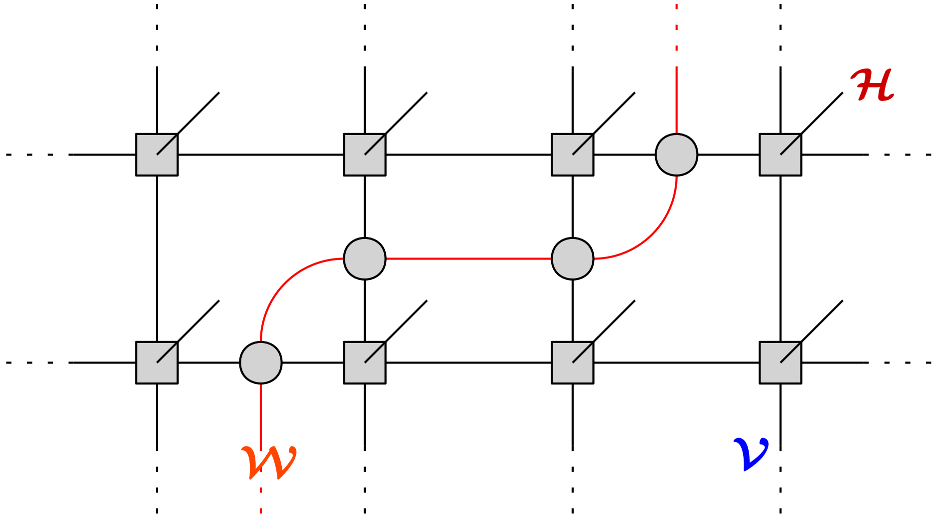

involving a further auxiliary space $\mathcal{W}$

and a further tensor $\boxed{B_{p,q}^{\alpha,\beta}}$

with $\alpha, \beta \in \{1, 2, \dots, \dim(\mathcal{W})\}$

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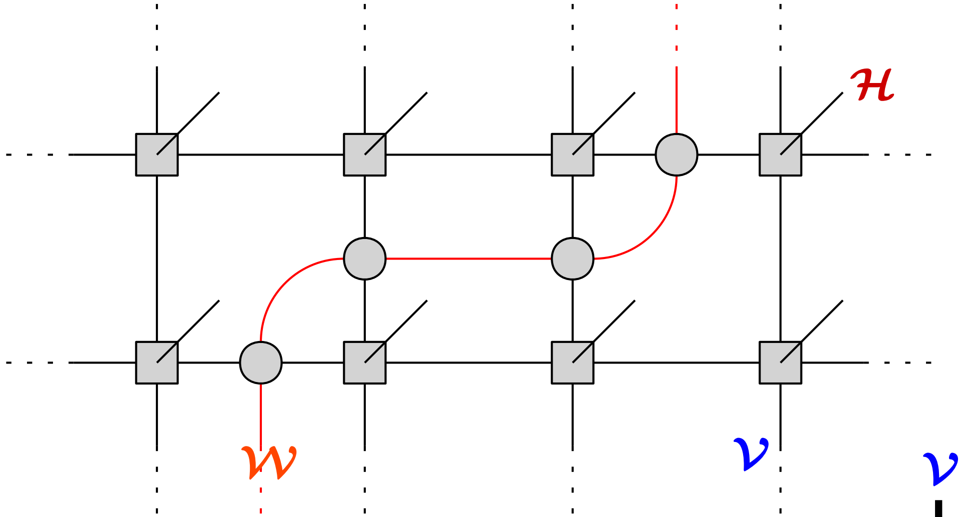

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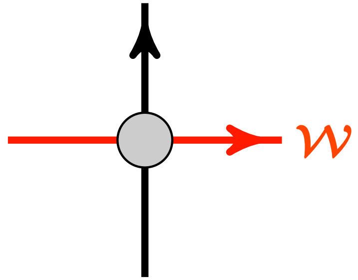

$\Longrightarrow$ **task** : formalization of line defects in 2-d systems

tool : categories and bicategories

Line defects

desirable **properties of line defects**

can carry point-like insertions
(*defect fields*)

can be fused

are oriented & can be deformed

duality amounts to orientation reversal

Line defects

desirable **properties of line defects**	$\longleftrightarrow$	**mathematical structure**
can carry point-like insertions (*defect fields*)		category $\mathcal{C}$ of defects
can be fused		monoidal structure on $\mathcal{C}$
are oriented & can be deformed		rigid dualities on $\mathcal{C}$
duality amounts to orientation reversal		pivotal structure on $\mathcal{C}$

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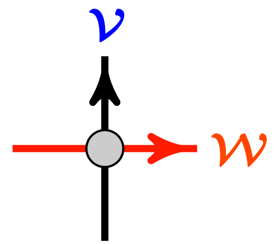
spherical fusion category $\mathcal{C}$ of defects

Bicategorical setup

Two monoidal categories

- *MPO* tensor $=$ 

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- thus linear map $B_{\mathcal{W}}(v):\mathcal{W}\longrightarrow\mathcal{W}$ for any $v\in\mathcal{V}\otimes\mathcal{V}$

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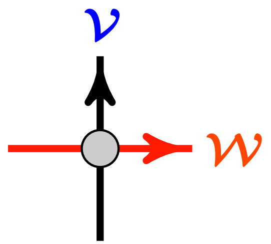
- *MPO* tensor $=$ 

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  with $I_{\mathcal{C}} = \{$ iso classes of simple $B_{\mathcal{W}}$-modules $\}$

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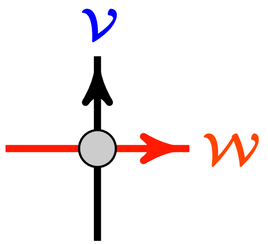
- *MPO* tensor $=$ 

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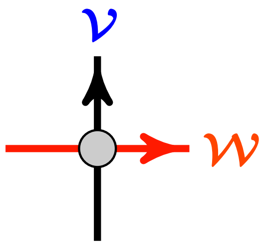

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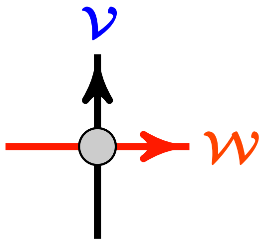
- *MPO* tensor $=$ 

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  $=\{$ iso classes of simple objects of $\mathcal{D}\,\}$

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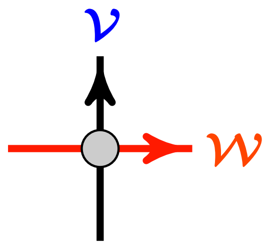
- *MPO* tensor = 

 + semisimplicity assumption

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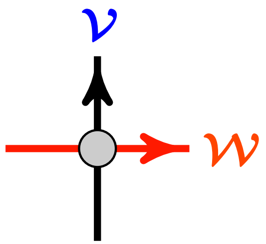

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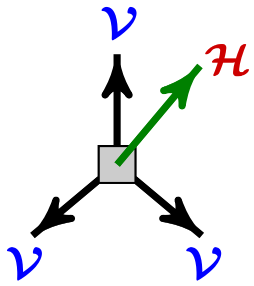

$\mathcal{V}^{\otimes 3} \otimes \mathcal{H} \longrightarrow \mathbb{C}$

specialized for convenience to hexagonal lattice

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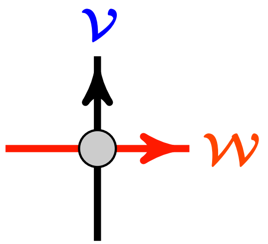
- *MPO* tensor $=$ 

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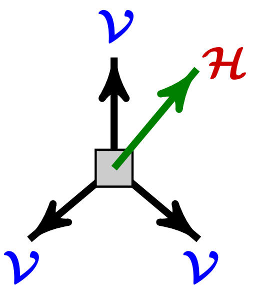
- PEPS tensor $=$ 

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specialized for convenience to hexagonal lattice

- fusion of defect lines $+$ concatenation of PEPS $+\ldots$

$\Longrightarrow$ $\mathcal{C}$ and $\mathcal{D}$ are *fusion* categories

Two monoidal categories

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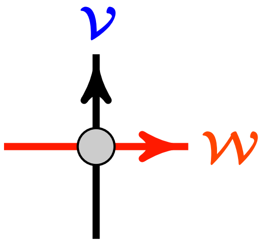

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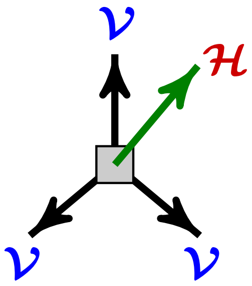

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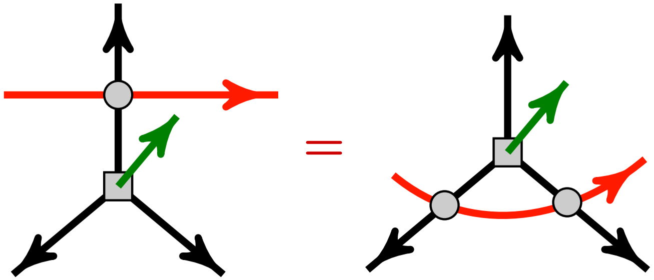
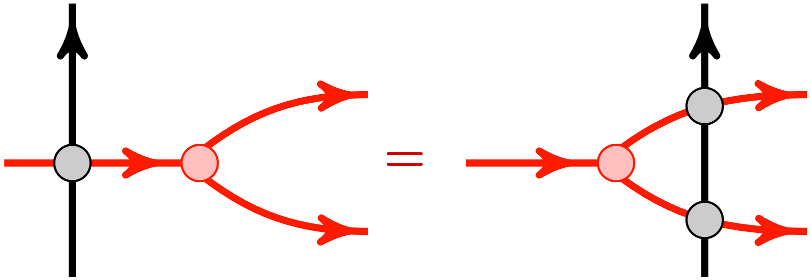
specialized for convenience to hexagonal lattice

- compatibility conditions :



(defect lines are *topological*)



(compatibility with fusion of defects)

Bicategory

- interpretation of compatibility conditions :

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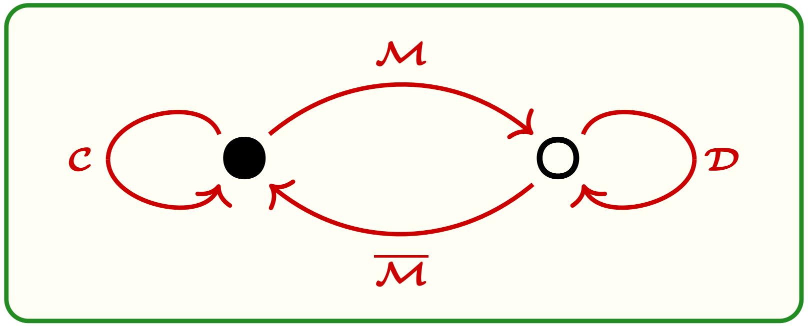

with some $\mathcal{C}$-$\mathcal{D}$-bimodule category $\mathcal{M}$

Bicategory

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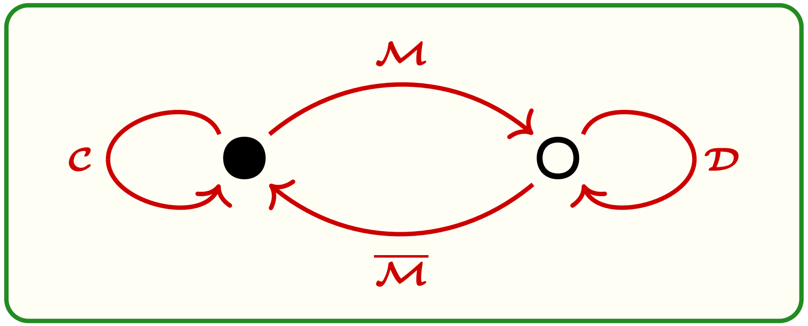

with some $\mathcal{C}$-$\mathcal{D}$-bimodule category $\mathcal{M}$

- minimal realization : $\mathcal{M}$ *invertible* bimodule

$\Longrightarrow$ 2-Morita context :

$\mathcal{D} = \mathcal{F}un_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ and $\overline{\mathcal{M}} = \mathcal{F}un_{\mathcal{C}}(\mathcal{M},\mathcal{C})$

Bicategory

- interpretation of compatibility conditions :

  pentagon identities for suitable mixed associators

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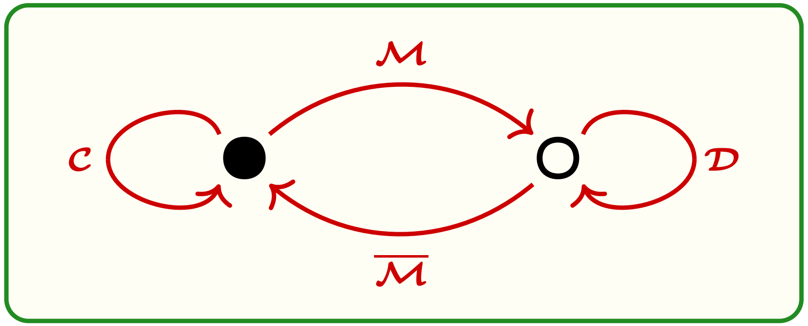

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- lattice interpretation of $\mathcal{M}$:

  labeling the 2-cells of the two-dimensional canvas on which the MPO lives

Lattice model

Lattice model

☞ lattice model on hexagonal lattice in the bicategorical setting :

- physical space $\mathcal{H} = \bigoplus_{\alpha,\beta,\gamma \in I_{\mathcal{D}}} \mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \beta, \gamma)$

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$\Longrightarrow$ 6j symbols $\longleftrightarrow$ tetrahedra

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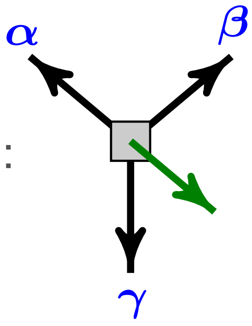

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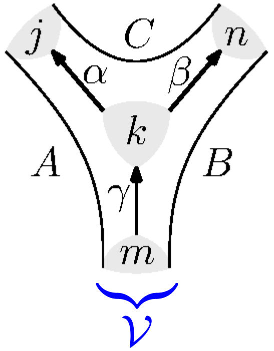

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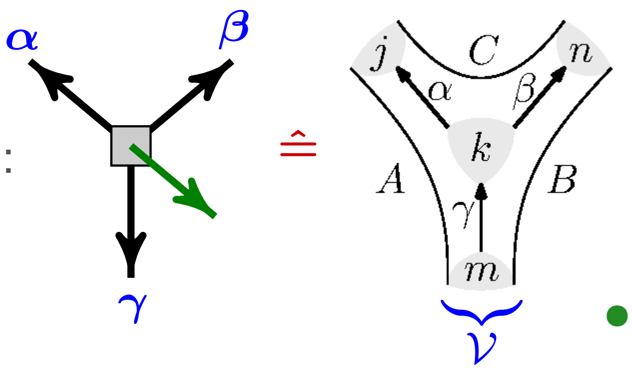
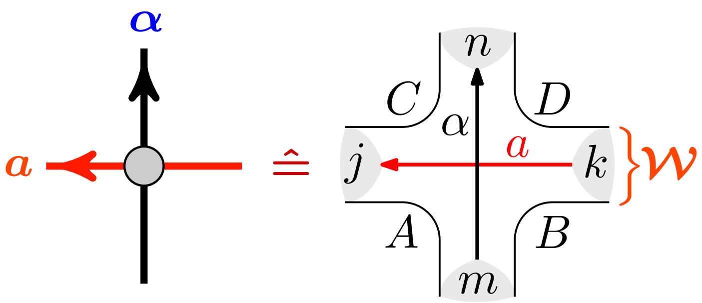


- MPO :



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- ${}^{0}\mathsf{F}$: 6j-symbols for $\mathcal{C}$ as monoidal category
- ${}^{4}\mathsf{F}$: 6j-symbols for $\mathcal{D}$ as monoidal category
- ${}^{1}\mathsf{F} =$ fusion of MPO tensors : mixed 6j-symbols for $\mathcal{M}$ as left $\mathcal{C}$-module
- ${}^{3}\mathsf{F} =$ PEPS : mixed 6j-symbols for $\mathcal{M}$ as right $\mathcal{D}$-module
- ${}^{2}\mathsf{F} =$ MPO : mixed 6j-symbols for $\mathcal{M}$ as bimodule

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$\Longrightarrow$ pentagon identities including e.g.

- $((a\otimes b)\triangleright A)\triangleleft\alpha \;\substack{\longrightarrow\;\longrightarrow\\ \longrightarrow\;\longrightarrow\;\longrightarrow}\; a\triangleright(b\triangleright(A\triangleleft\alpha))$

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Lattice model – main insights

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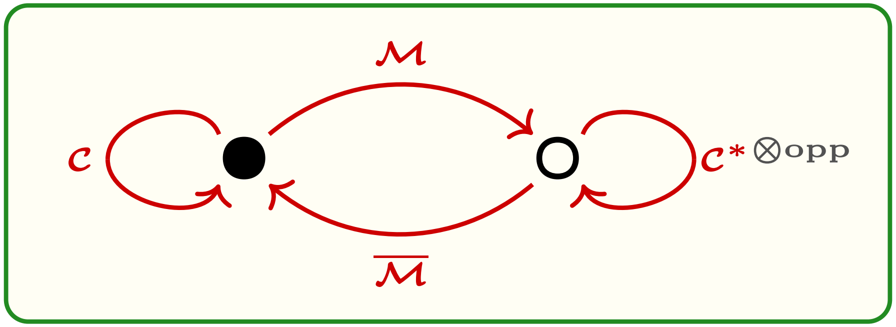
☞ PEPS in 2 dimensions and their MPO symmetries fit into a strong 2-Morita context

$$\begin{tikzcd}
\bullet \arrow[loop left, "\mathcal{C}"] \arrow[r, bend left, "\mathcal{M}"] & \circ \arrow[loop right, "\mathcal{C}^{*} \otimes \mathrm{opp}"] \arrow[l, bend left, "\overline{\mathcal{M}}"]
\end{tikzcd}$$

Lattice model – main insights

- PEPS in 2 dimensions and their MPO symmetries fit into a strong 2-Morita context



- contracting a network of PEPS and MPO tensors gives candidate ground state as element of a subspace of $\mathcal{H}^{\otimes N}$
- any indecomposable pivotal $\mathcal{C}$-module $\mathcal{M}$ gives a PEPS
- the PEPS for $\mathcal{M}$ exhibits MPO symmetries given by $\mathcal{C}^* = \mathcal{F}un_{\mathcal{C}}(\mathcal{M},\mathcal{M})$
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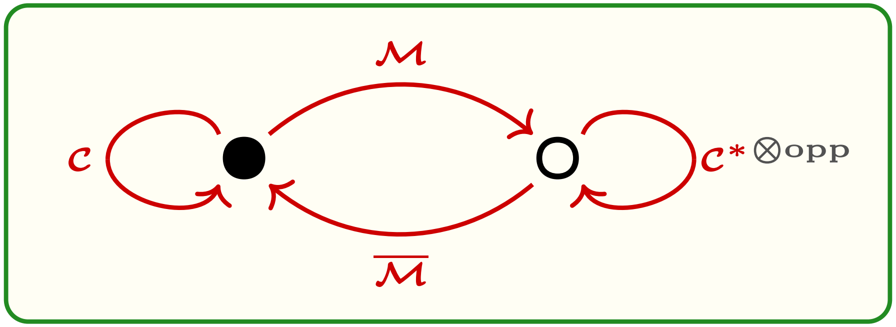

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- ☞ pivotal categorical Morita equivalence relates dual descriptions :

  $\mathcal{D} = \mathcal{F}un_{\mathcal{C}}(\mathcal{M},\mathcal{M}) \qquad \mathcal{C} = \mathcal{F}un_{\mathcal{D}}(\mathcal{M},\mathcal{M}) \qquad \mathcal{Z}(\mathcal{C}) = \mathcal{Z}(\mathcal{D})$

2-Morita contexts

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2-Morita context :

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- finite multi tensor categories $\mathcal{C}$ and $\mathcal{D}$
- bimodules ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ and ${}_{\mathcal{D}}\mathcal{N}_{\mathcal{C}}$
- bimodule functors $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \longrightarrow \mathcal{C}$ and $\mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M} \longrightarrow \mathcal{D}$
- two bimodule natural transformations filling a diagram for those functors
- pentagon identities for these natural transformations

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- composition of 1-morphisms via $\otimes_{\mathcal{C}}$, $\otimes_{\mathcal{D}}$, $\triangleleft_{\mathcal{C}}$, $\triangleright_{\mathcal{C}}$, $\triangleleft_{\mathcal{D}}$, $\triangleright_{\mathcal{D}}$, $\boxdot_{\mathcal{C}}$, $\boxdot_{\mathcal{D}}$
- 16 associativity constraints
- 32 pentagon diagrams “commuting by construction”

2-Morita contexts

Proposition :

- any exact module category $\mathcal{M}$ over a finite tensor category $\mathcal{C}$ gives a strong Morita context
- this admits rigid dualities :

$m^\vee = \underline{\mathbf{Hom}}_{\mathcal{M}}(m,-)$ internal Hom

${}^\vee m = \underline{\mathbf{coHom}}_{\mathcal{M}}(m,-)$ internal coHom

for $m \in \mathcal{M}$

and similarly for $\overline{\mathcal{M}}$

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☞ double duals given by relative Serre functors : $m^{\vee\vee} = \mathbf{S}^{\mathrm{r}}_{\mathcal{M}}(m)$

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☞ pivotal structure on an exact module category $\mathcal{M}$ over a pivotal tensor category :

natural isomorphism $\mathrm{id}_{\mathcal{M}} \xrightarrow{\cong} \mathbf{S}^{\mathrm{r}}_{\mathcal{M}}$

SCHAUMANN, SHIMIZU

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☞ double duals given by relative Serre functors : $m^{\vee\vee} = \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m)$

☞ pivotal structure on a rigid bicategory $\mathcal{B}$:
pseudonatural equivalence $\boxed{\pi : \mathrm{id}_{\mathcal{B}} \xrightarrow{\simeq} (-)^{\vee\vee}}$

☞ spherical 2-Morita context for a spherical module category $\mathcal{M}$ over a unimodular (Radford-) spherical finite tensor category :
π^{2} coinciding with another pseudonatural equivalence $\mathrm{id} \longrightarrow (-)^{\vee\vee\vee\vee}$
involving the relative Serre pseudofunctor $\mathbb{S}$ on the associated bicategory $\mathcal{B}_{\mathcal{M}}$

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- sphericality of the 2-Morita context captures preservation of sphericality under pivotal 2-Morita equivalence

State-sum models

State-sum TFTs

goal : handle on more general lattice models including defects

tool : state-sum TFTs

TURAEV-VIRO

BARRETT-WESTBURY

TURAEV-VIRELIZIER

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TURAEV-VIRO
BARRETT-WESTBURY
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T-V TFT :

- input data : spherical fusion category $\mathcal{D}$
  oriented three-manifold $\mathcal{M}$ possibly with gluing boundary Σ
  skeleton Δ for $\mathcal{M}$ (generalized triangulation , $\Delta_0 \cap \Sigma = \emptyset$)

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- state sum variables : simple object $d \in \mathcal{D}$ for each 2-cell in $\Delta_2 \backslash \Sigma$
- vector spaces $V_{e_\pm}$: for each half edge (edge $e \in \Delta_1$ + end point)

$$V_{e_+} = \mathrm{Hom}_{\mathcal{D}}(1, d_1 \otimes \cdots \otimes d_n) \cong (V_{e_-})^*$$

(d_i state sum variables of adjacent 2-cells)

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  $$V_{e_+} = \mathrm{Hom}_{\mathcal{D}}(1, d_1 \otimes \cdots \otimes d_n) \cong (V_{e\_})^*$$
- vector space V_Δ by tensoring $V_{e_\pm}$'s
- apply evaluation maps obtained from graphical calculus on spheres that surround vertices in $\Delta_0 \backslash \Sigma$ to canonical vector in $V_\Delta \otimes V_\Delta^*$ and sum over state sum variables

$\Longrightarrow$ TFT state space $\mathrm{tft}_{\mathcal{D}}(\Sigma)$ independent of the choice of Δ

State-sum models for PEPS

special case : three-manifold $M = M_{\Sigma} := \Sigma \times [0,1]$ with

- gluing boundary $\Sigma \times \{1\}$ with state space $\mathrm{tft}_{\mathcal{D}}(\Sigma) =: \mathcal{H}_{\Sigma}^{0}$
- physical boundary $\Sigma \times \{0\}$ with specified boundary condition $\mathcal{M}$

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i.e. $\mathcal{M}$ a $\mathcal{D}$-module category

KITAEV-KONG

JF-SCHWEIGERT-VALENTINO

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- consider the image of $1 \in \mathbb{C} = \mathrm{tft}_{\mathcal{D}}(\emptyset)$
  under $\mathrm{tft}_{\mathcal{D}}(M_\Sigma) \colon \mathbb{C} \longrightarrow \mathrm{tft}_{\mathcal{D}}(\Sigma)$
- interpret this vector in $\mathrm{tft}_{\mathcal{D}}(\Sigma)$ as a state described by a PEPS tensor

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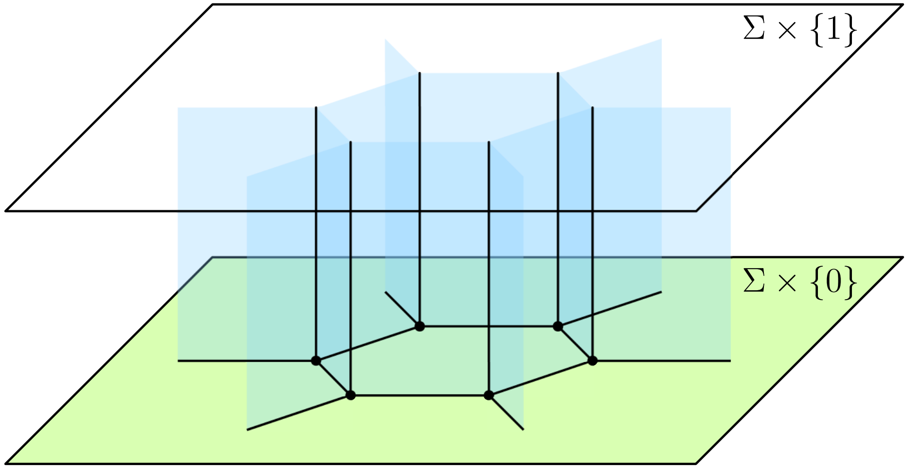

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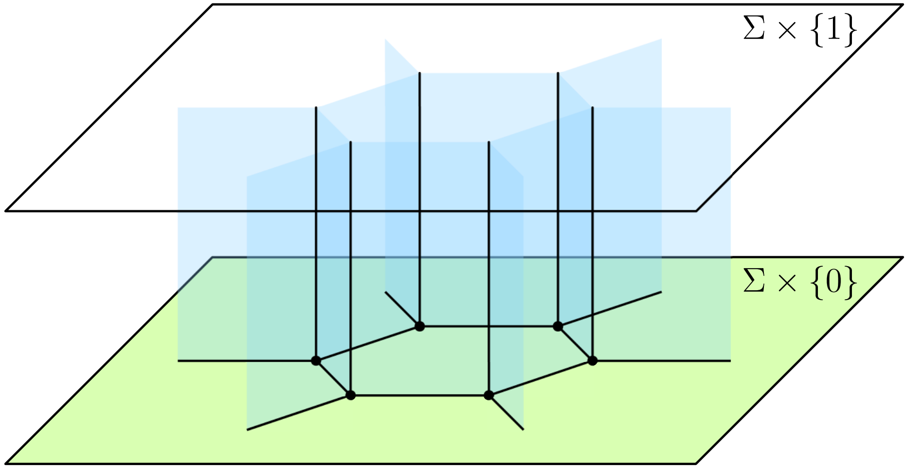

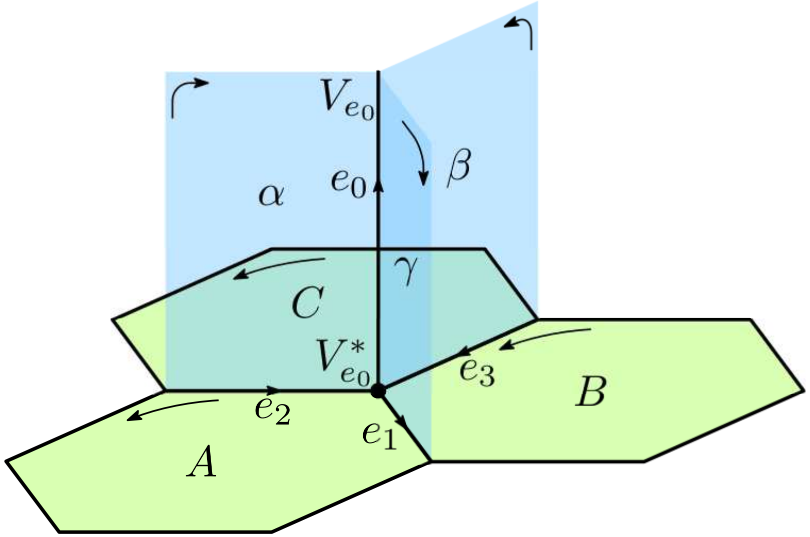

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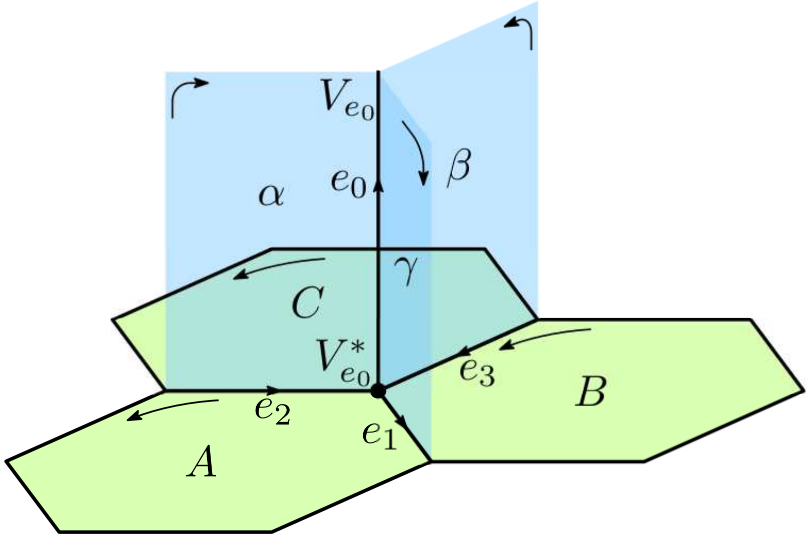

- state sum variables :

  $\alpha \in \mathcal{D}$ for faces in the interior of M_Σ

  $A \in \mathcal{M}$ for faces on $\Sigma \times \{0\}$

- vector spaces of invariants :

  Hom spaces in $\mathcal{D}$ for edges in the interior

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State-sum models for PEPS

special case : three-manifold $M = M_\Sigma := \Sigma \times [0,1]$ with

- gluing boundary $\Sigma \times \{1\}$ with state space $\mathrm{tft}_{\mathcal{D}}(\Sigma) =: \mathcal{H}_\Sigma^0$
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main idea :

- consider the image of $1 \in \mathbb{C} = \mathrm{tft}_{\mathcal{D}}(\emptyset)$
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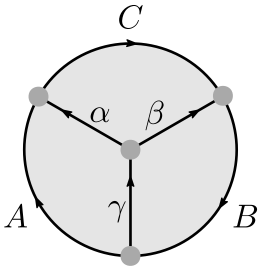
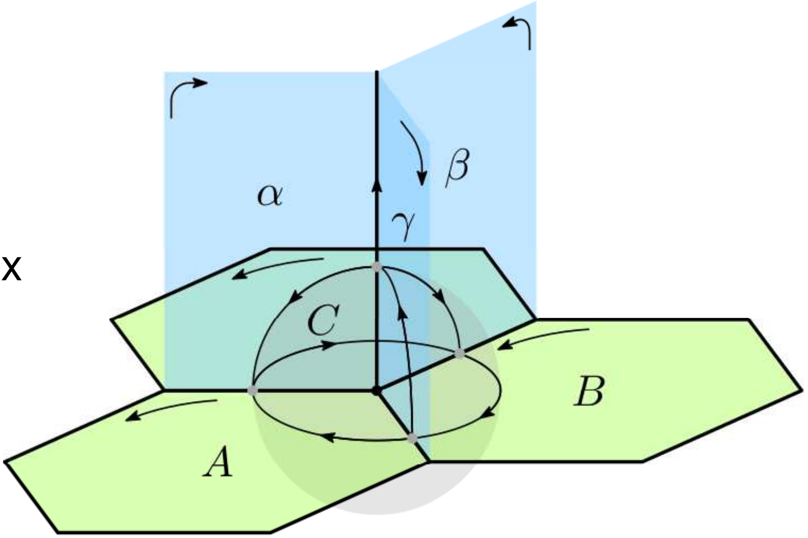
$\Longrightarrow$ 6j-symbol





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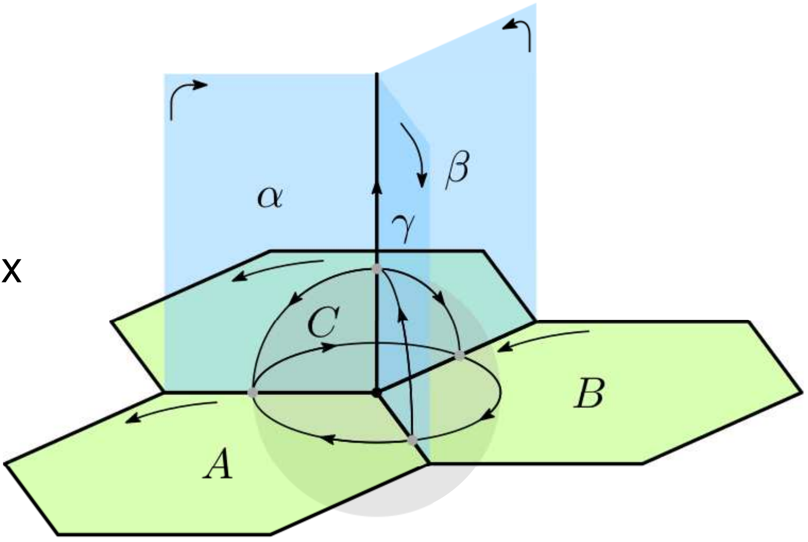
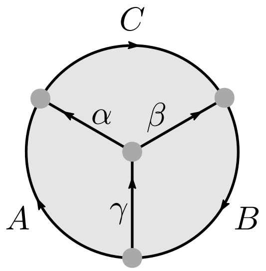


$\Longrightarrow$ 6j-symbol  $= \mathbf{PEPS}_{\mathcal{D},\mathcal{M}}$

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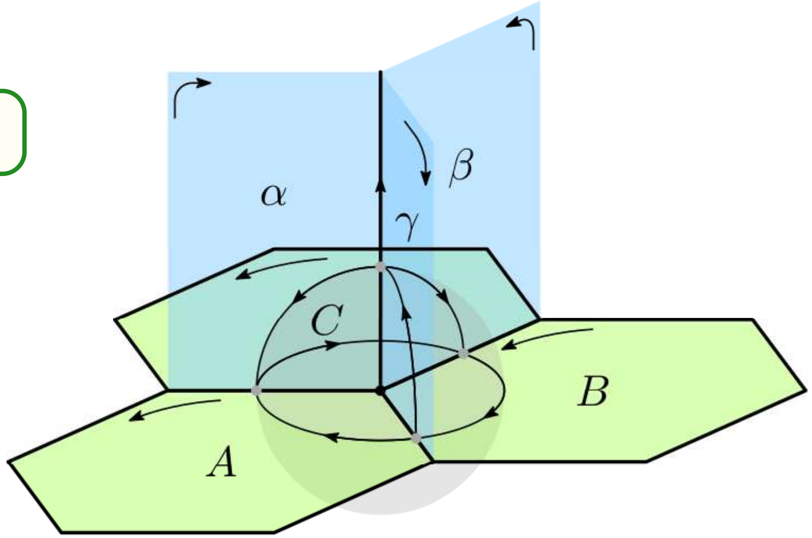

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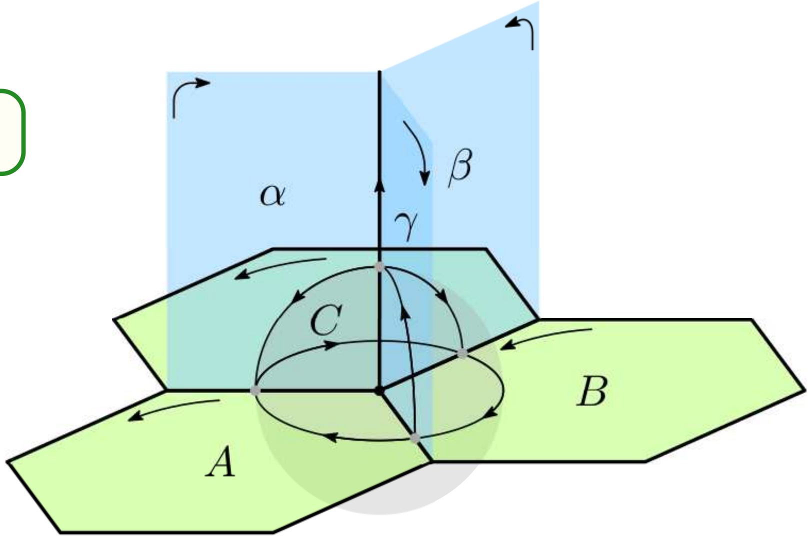
state-sum TFT provides
a holographic description of PEPS
that is independent of lattices



Generalizations

further issues :

- inclusion of MPO symmetries :
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- can be analyzed numerically after obtaining 6j-symbols by solving pentagons

  LOOTENS . . . . . .
- case $\mathcal{M} \simeq A$-mod for A pointed considered earlier

  LUO-LAKE-WU

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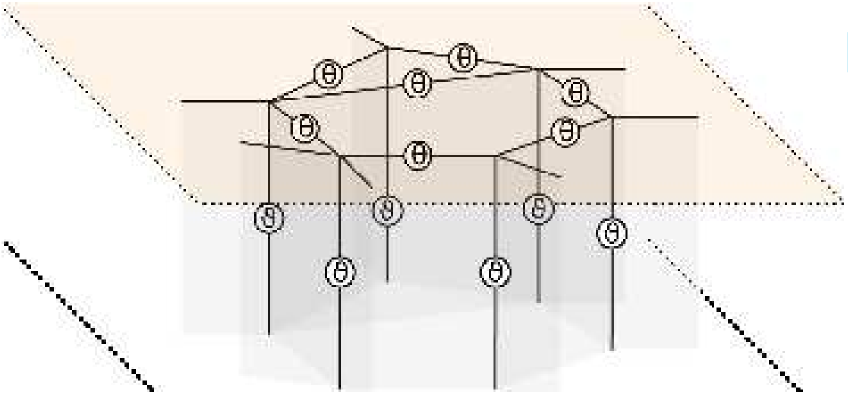
LOOTENS . . . . . .

generalizations :

- include non-trivial dynamics by sandwiching with additional physical boundary

e.g for generalized
Ising models



DELCAMP-ISHTIAQUE

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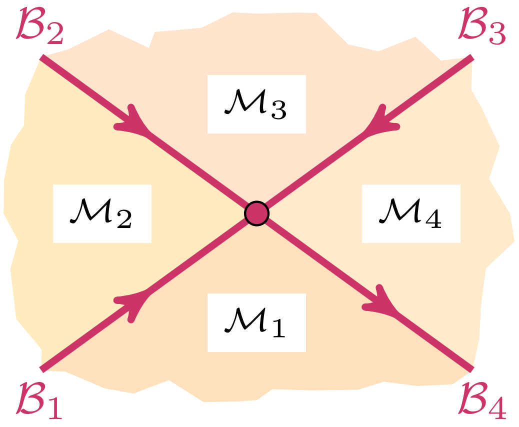

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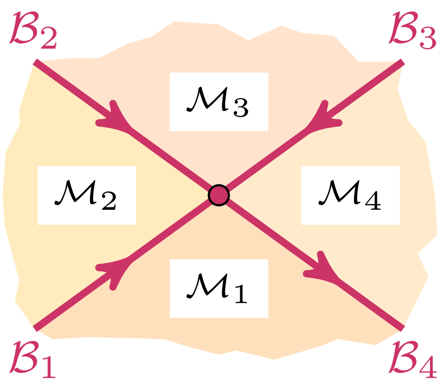
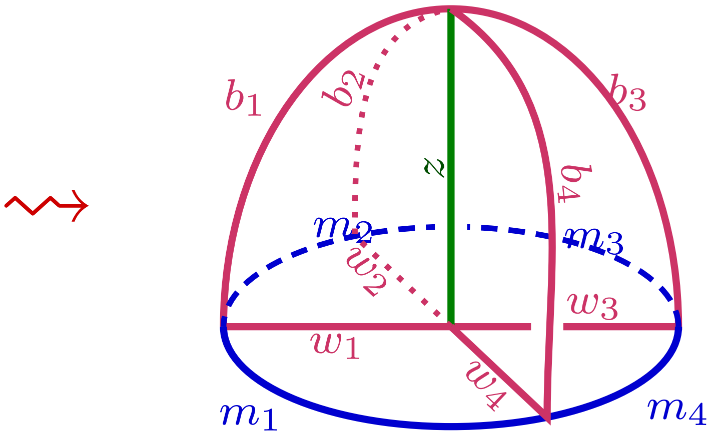
  important tool : calculus of *extruded graphs*

FARNSTEINER-SCHWEIGERT





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further directions :

- non-semisimple models

  (results for pivotal bicategories obtained in framework of finite tensor categories)
- non-rigid dualities
- higher dimensions (higher fusion categories)

THANK YOU