On Topological Two-Parameter Quantum Groups: The Universal R-matrix and Central Elements

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Outline

Introduction and Motivation

Key Developments on Two-Parameter Quantum Groups

- Systematic construction and weight module theory for finite and affine types e.g., [BW04,BGH07,HRZ08]
- Explicit basic R-matrices and their applications to FRT-reconstruction and RLL-realization e.g., [JL14,HXZ24,ZHJ24,ZHX24]

[BW04] Benkart, G., Witherspoon, S. (2004) *Two-parameter quantum groups and Drinfel'd doubles*. Algebras Represent. Theory 7(3), 261–286.
[BGH07] Bergeron, N., Gao, Y., Hu, N. (2007) *Representations of two-parameter quantum orthogonal groups and symplectic groups*. In: AMS/IP Stud. Adv. Math., 39, Amer. Math. Soc., Providence, RI, 1–21.

[HRZ08] Hu, N., Rosso, M., Zhang, H. (2008) *Two-parameter quantum affine algebra $U_{r,s}(\widehat{\mathfrak{sl}_n})$, Drinfel'd realization and quantum affine Lyndon basis*. Commun. Math. Phys. 278(2), 453–486.

[JL14] Jing, N., Liu, M. (2014) *R-matrix realization of two-parameter quantum group $U_{r,s}(\mathfrak{gl}_n)$*. Commun. Math. Stat. 2, 211–230.
[HXZ24] Hu, N., Xu, X., Zhuang, R. (2024) *RLL-realization of two-parameter quantum affine algebra of type $B_n^{(1)}$ and regulated quantum Lyndon bases*. arXiv:2405.06587.
[ZHJ24] Zhong, X., Hu, N., Jing, N. (2026) *RLL-realization of two-parameter quantum affine algebra of type $C_n^{(1)}$*. J. Algebra Appl. (to appear).
[ZHX24] Zhuang, R., Hu, N., Xu, X. (2024) *RLL-realization of two-parameter quantum affine algebra of type $D_n^{(1)}$*. Pacific J. Math. 329(2), 357–395.

Motivation

- **Open Problem**: the existence and explicit construction of the *universal R-matrix* for all two-parameter quantum groups
- Universal R-matrix is a prerequisite for formulating isomorphism theorems:

$$\text{FRT} \overset{\text{finite}}{\longleftrightarrow} \text{Chevalley}, \qquad \text{RLL} \overset{\text{affine}}{\longleftrightarrow} \text{Drinfeld}$$

- Topological framework is necessary for dual basis in Cartan part

[KT00] Kassel, C., Turaev, V. (2000) *Biquantization of Lie bialgebras*. Pacific J. Math. 195(2), 297–369.

Main Results

Main Contributions

1. Introduced a topological two-parameter quantum group

$$U_{h,h'}(\mathfrak{g}),\ \ r = e^{h}, s = e^{h'}, \omega_i = e^{hH_i}, \omega_i' = e^{h'H_i'}$$

   inspired by Drinfeld's quantum double construction

2. **Explicit universal R-matrix** for $U_{h,h'}(\mathfrak{g})$

$$\mathcal{R} = \left(\prod_{\alpha\in\Phi^+}^{\succ} \exp_{\langle \omega_\alpha', \omega_\alpha \rangle} \frac{\mathcal{E}_\alpha \otimes \mathcal{F}_\alpha}{\langle \mathcal{F}_\alpha, \mathcal{E}_\alpha \rangle} \right) \exp\left(hh' \sum_{i,j=1}^{n} \left[(\ln A)^{-1} \right]_{ij} H_i \otimes H_j' \right)$$

3. Uniformly recovered all known basic R-matrices of classical types (recently construct one for type G_2)

4. Central elements $c_\lambda = \mathrm{Tr}_2 \left((\mathcal{R}^+)^{\mathrm{op}} \mathcal{K}_\lambda' \mathcal{R}^+ \mathcal{K}_\lambda (1 \otimes \Theta) \right)$ and $c_\lambda = z_\lambda$, the unique one $\langle z_\lambda \mid - \rangle = \mathrm{tr}_{L(\lambda)}(- \circ \Theta)$, refining the Harish-Chandra theory

Preliminaries: Basic Notations

Φ, Π	Root system and simple root set of simple Lie algebra $\mathfrak{g}$
$C = (c_{ij})$	Cartan matrix of $\mathfrak{g}$
Q, P	Root lattice and weight lattice
$\langle \cdot, \cdot \rangle$	Euler form of $\mathfrak{g}$
m	$\det(C) = \min\{k \in \mathbb{Z}^+ \mid kP \subseteq Q\}$
r, s	Two indeterminates
$\mathbb{K} \supset \mathbb{Q}(r, s)$	A field of characteristic zero containing $r^{\frac{1}{m}}, s^{\frac{1}{m}}$
$a_{ij} = r^{\langle i,j \rangle} s^{-\langle j,i \rangle}$	Structure constant of $U_{r,s}(\mathfrak{g})$
$A = (a_{ij})$	Structure constant matrix

Here, the Euler form of $\mathfrak{g}$ is the bilinear form $\langle -, - \rangle$ defined on Q satisfying

$$\langle i, j \rangle := \langle \alpha_i, \alpha_j \rangle = \begin{cases} d_i c_{ij} & i < j, \\ d_i & i = j, \\ 0 & i > j. \end{cases}$$

For type D, we revise $\langle n-1, n \rangle = -1$, $\langle n, n-1 \rangle = 1$ as in [BGH06, HP12]. It naturally extends to the weight lattice P such that

$$\langle \varpi_i, \varpi_j \rangle := \sum_{k,l=1}^{n} c^{ki} c^{lj} \langle k, l \rangle$$

[BGH06] Bergeron, N., Gao, Y., Hu, N. (2006) *Drinfel'd doubles and Lusztig's symmetries of two-parameter quantum groups*. J. Algebra 301(1), 378–405.

[HP12] Hu, N., Pei, Y. (2012) *Notes on two-parameter groups (II)*. Comm. Algebra 40(9), 3202–3220.

Definition of $U_{r,s}(\mathfrak{g})$

Hopf Algebra $U_{r,s}(\mathfrak{g})$ (Generators: $e_i, f_i, \omega_i^{\pm 1}, \omega_i'^{\pm 1}$)

- (A1) $\{\omega_i, \omega_i'\}$ are invertible and pairwise commute
- (A2) $\omega_i e_j \omega_i^{-1} = a_{ji} e_j, \qquad \omega_i' e_j \omega_i'^{-1} = a_{ij}^{-1} e_j,$
  $\omega_i f_j \omega_i^{-1} = a_{ji}{}^{-1} f_j, \quad \omega_i' f_j \omega_i'^{-1} = a_{ij} f_j$
- (A3) $[e_i, f_j] = \delta_{ij} \frac{\omega_i - \omega_i'}{r_i - s_i}$
- (A4) $(\mathrm{ad}_l\, e_i)^{1-c_{ij}}(e_j) = 0, \qquad (\mathrm{ad}_r\, f_i)^{1-c_{ij}}(f_j) = 0, \qquad (i \neq j)$
- (C1) $\{\omega_i, \omega_i'\}$ are group-like
- (C2) e_i is $(1, \omega_i)$-skew primitive, f_i is $(\omega_i', 1)$-skew primitive

Skew-dual pairing $\langle -, - \rangle : U_{r,s}(\mathfrak{b}^-) \times U_{r,s}(\mathfrak{b}^+) \to \mathbb{K}$

$$\langle f_i, e_j \rangle = \delta_{ij} \frac{1}{s_i - r_i}, \qquad \langle \omega_i', \omega_j \rangle = a_{ij},$$

$$\langle \omega_i'^{\tau_i}, \omega_j^{\tau_j} \rangle = \langle \omega_i', \omega_j \rangle^{\tau_i \tau_j} \qquad (\ \tau_i = \pm 1\ )$$

Definition of $U_{h,h'}(\mathfrak{g})$

Topological Hopf Algebra $U_{h,h'}(\mathfrak{g})$ (Generators: e_i, f_i, H_i, H'_i)

- Over a field $\widehat{\mathbb{K}} \supset \mathbb{Q}[[h, h']]$
- $r = e^h, s = e^{h'}; \omega_i = e^{hH_i}, \omega'_i = e^{h'H'_i}$
- $(A1)$ $\{H_i, H'_i\}$ are pairwise commute
- $(A2)$ $[H_i, e_j] = h^{-1}\ln(a_{ji})e_j, \qquad [H'_i, e_j] = -h'^{-1}\ln(a_{ij})e_j$
  $[H_i, f_j] = -h^{-1}\ln(a_{ji})f_j, \quad [H'_i, f_j] = h'^{-1}\ln(a_{ij})f_j$
- $(A3)$ $[e_i, f_j] = \delta_{ij}\frac{e^{d_iH_i} - e^{d_iH'_i}}{e^{d_ih} - e^{d_ih'}} = \delta_{ij}\, e^{\frac{hH_i + h'H'_i - (h+h')d_i}{2}}\, \frac{\sinh \frac{hH_i - h'H'_i}{2}}{\sinh \frac{(h-h')d_i}{2}}$
- $(A4)$ $(\mathrm{ad}_l\, e_i)^{1-c_{ij}}(e_j) = 0, \qquad (\mathrm{ad}_r\, f_i)^{1-c_{ij}}(f_j) = 0, \qquad (i \neq j)$
- (C1) $\{H_i, H'_i\}$ are primitive
- (C2) e_i is $(1, e^{hH_i})$-skew primitive, f_i is $(e^{h'H'_i}, 1)$-skew primitive

$$U_{r,s}(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g}): \quad r \to q,\ s \to q^{-1},\ \omega_i' = \omega_i^{-1}$$
$$U_{h,h'}(\mathfrak{g}) \longrightarrow U_\hbar(\mathfrak{g}): \quad h \to \hbar,\ h' \to -\hbar,\ H_i = H_i'$$

Extended skew-dual pairing $\langle -, - \rangle : U_{h,h'}(\mathfrak{b}^-) \times U_{h,h'}(\mathfrak{b}^+) \longrightarrow \widehat{\mathbb{K}}$

$$\langle f_i, e_j \rangle = \delta_{ij}\frac{1}{s_i - r_i} = \delta_{ij}\frac{1}{e^{d_i h'} - e^{d_i h}} \qquad 1 \leqslant i, j \leqslant n,$$
$$\begin{aligned} \langle H_i', H_j \rangle &= (hh')^{-1} \ln(\langle \omega_i', \omega_j \rangle) \\ &= (hh')^{-1}(h\langle i,j\rangle - h'\langle j,i\rangle), \qquad 1 \leqslant i, j \leqslant n, \end{aligned}$$

Weight Modules and Convex PBW-Type Lyndon Basis

Weight Modules

- Verma module $M(\lambda)$ and irreducible quotient $L(\lambda)$
- Weight space decomposition: $L(\lambda) = \bigoplus_{\eta \leq \lambda} L(\lambda)_\eta$
- Type σ weight space: $M_{\eta,\sigma}$ (trivial σ = type 1)

Convex PBW-Type Basis

- Standard Lyndon words $\leftrightarrow$ positive roots
- Quantum root vectors $\mathcal{E}_\gamma := [\mathcal{E}_\alpha, \mathcal{E}_\beta]_{\langle \omega'_\beta, \omega_\beta \rangle}$ inductively from costandard decomposition (α, β) of γ
- Basis of U^+: $\{\prod^{\succ}_{\alpha \in \Phi^+} \mathcal{E}_\alpha^{n_\alpha} \mid n_\alpha \in \mathbb{N}\}$
- Basis of U^-: $\{\prod^{\prec}_{\alpha \in \Phi^+} \mathcal{F}_\alpha^{n_\alpha} \mid n_\alpha \in \mathbb{N}\}$

Weight Modules

When $\lambda \in P$, denote $M(\lambda) := M(\varrho^\lambda) = U_{r,s} \otimes_{\mathcal{B}} \varrho^\lambda$, and $L(\lambda)$ the unique irreducible quotient of $M(\lambda)$.

$$\text{(For } U_{r,s}(\mathfrak{g}) \text{)}\ \varrho^\lambda(\omega_i) := \prod_{j=1}^{n} \langle \omega_j', \omega_i \rangle^{\lambda_j}, \qquad \varrho^\lambda(\omega_i') := \prod_{j=1}^{n} \langle \omega_i', \omega_j \rangle^{-\lambda_j};$$

$$\text{(For } U_{h,h'}(\mathfrak{g}) \text{)}\ \varrho^\lambda(H_i) := h^{-1} \sum_{j=1}^{n} \lambda_j \ln a_{ji}, \quad \varrho^\lambda(H_i') := -h'^{-1} \sum_{j=1}^{n} \lambda_j \ln a_{ij}.$$

Constructions for $U_{r,s}(\mathfrak{g})$ [BW04,BGH07,PHR10] which could be naturally extended to the version of $U_{h,h'}(\mathfrak{g})$.

[PHR10] Pei, Y., Hu, N., Rosso, M. (2010) *Multi-parameter quantum groups and quantum shuffles (I)*. Contemp. Math., 506, Amer. Math. Soc., Providence, RI, 145–171.

Lemma [BW04,BGH07,PHR10]

Let v_λ be a highest weight vector of $M(\lambda)$ for $\lambda \in P^+$. Then

$$L(\lambda) = M(\lambda) \big/ \big(\sum_{i=1}^{n} U f_i^{(\lambda, \alpha_i^\vee)+1} \cdot v_\lambda \big).$$

Also, it has the decomposition of weight space $L(\lambda) = \bigoplus_{\eta \leqslant \lambda} L(\lambda)_\eta$, where

$$\begin{aligned} L(\lambda)_\eta = \{ x \in L(\lambda) \mid \omega_i.x = \varrho^\eta(\omega_i)x = \langle \omega'_\eta, \omega_i \rangle x, \\ \omega'_i.x = \varrho^\eta(\omega'_i)x = \langle \omega'_i, \omega_\eta \rangle^{-1} x,\ 1 \leqslant i \leqslant n \}. \end{aligned}$$

σ-type weight space

For each $U_{r,s}(\mathfrak{g})$-mod M, each weight $\eta \in P$ and each group homomorphisms $\sigma : (\mathbb{Z}\Phi, +) \longrightarrow (\mathbb{K}^{\times}, \cdot)$, define σ-type weight space

$$M_{\eta,\sigma} := \{m \in M \mid \omega_i.x = \sigma(\alpha_i)\varrho^{\eta}(\omega_i)x,\ \omega_i'.x = \sigma(\alpha_i)\varrho^{\eta}(\omega_i')x,\ 1 \leqslant i \leqslant n\}.$$

Notice that $M = \bigoplus_{\sigma} M^{\sigma} := \bigoplus_{\sigma} \bigoplus_{\eta \in P} M_{\eta,\sigma}$. If $M = M^{\sigma}$, we say that M is of type σ. In particular, if σ is trivial, we say it is of type 1.

The convex PBW-type basis

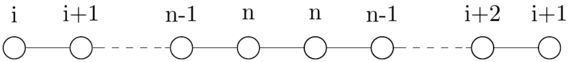
We recall the combinatorial construction of the convex PBW-type basis for the two-parameter quantum group $U_{r,s}(\mathfrak{g})$ (e.g., [BH08, CHW23, HXZ24])

[BH08] Bai, X., Hu, N. (2008) *Two-parameter quantum groups of exceptional type E-series and convex PBW-type basis*. Algebra Colloq. 15(4), 619–636.
[CHW23] Chen, X., Hu, N., Wang, X. (2023) *Convex PBW-type Lyndon bases and restricted two-parameter quantum group of type F_4*. Acta Math. Sin.-Engl. Ser. 39, 1053–1084.
[HXZ24] Hu, N., Xu, X., Zhuang, R. (2024) *RLL-realization of two-parameter quantum affine algebra of type $B_n^{(1)}$ and regulated quantum Lyndon bases*. arXiv:2405.06587.

$\mathcal{A}$	the ordered alphabet set $\{1, 2, \cdots, n\}$ with $1 < 2 < \cdots < n$.
$\mathcal{A}^*$	the set of all words in A with the induced lexicographical order $<$
Lyndon word l	a word $l \in A^*$, s.t. $l <$ all its proper right factors
Lyndon decomposition	a factorization $l = uv$ into two proper Lyndon words (u, v)
(Co)standard pair of l	the Lyndon decomposition $l = uv$, where u (resp. v) is the shortest Lyndon word that is a proper left factor of l
ℓ	Lalonde-Ram correspondence: $\Phi^+ \leftrightarrow$ standard Lyndon words in $\mathcal{A}^*$ $\alpha_i \leftrightarrow \ell(\alpha_i) = i$
$\prec$	the convex ordering $\prec$ on Φ^+ induced by ℓ
(Co)standard pair of γ	the pair (α, β) in Φ^+, s.t. $\gamma = \alpha + \beta$, $\ell(\gamma) = \ell(\alpha)\ell(\beta)$ is (co)standard

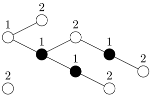
[LR95] Lalonde, M., Ram, A. (1995) *Standard Lyndon bases of Lie algebras and enveloping algebras*. Trans. Amer. Math. Soc. 347(5), 1821–1830.

Convex Ordering

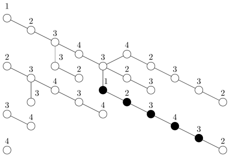


Type B_n

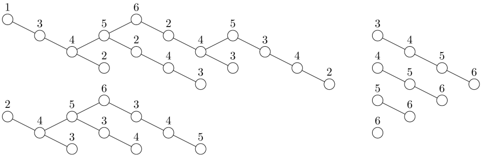
$$
\begin{aligned}
&\alpha_{1,2} \prec \alpha_{1,3} \prec \cdots \prec \alpha_{1,n} \prec \alpha_{1,n+1} \prec \alpha_{1,n'} \prec \cdots \prec \alpha_{1,3'} \prec \alpha_{1,2'} \\
&\prec \alpha_{2,3} \prec \cdots \prec \alpha_{2,n} \prec \alpha_{2,n+1} \prec \alpha_{2,n'} \prec \cdots \prec \alpha_{2,3'} \\
&\qquad \ddots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \iddots \\
&\prec \alpha_{n-1,n} \prec \alpha_{n-1,n+1} \prec \alpha_{n-1,n'} \\
&\prec \alpha_{n,n+1}.
\end{aligned}
$$



Type G_2



Type F_4



Type E_6

Write $\mathcal{E}_{\alpha_i} = e_i,\ \mathcal{F}_{\alpha_i} = f_i\ (1 \leqslant i \leqslant n)$. For each non-simple positive root $\gamma \in \Phi^+$ with its costandard pair (α, β) (i.e., the unique pair (α, β) of positive roots satisfying $\gamma = \alpha + \beta$ and $\alpha \prec \gamma \prec \beta$, where α is maximal), the quantum root vectors $\mathcal{E}_\gamma$ and $\mathcal{F}_\gamma$ can be defined through (r, s)-bracket $[-,-]_{(\cdot)}$ inductively:

$$\begin{aligned}
\mathcal{E}_\gamma &:= [\mathcal{E}_\alpha, \mathcal{E}_\beta]_{\langle \omega'_\beta, \omega_\alpha \rangle} &&= \mathcal{E}_\alpha \mathcal{E}_\beta - \langle \omega'_\beta, \omega_\alpha \rangle \mathcal{E}_\beta \mathcal{E}_\alpha, \\
\mathcal{F}_\gamma &:= [\mathcal{F}_\beta, \mathcal{F}_\alpha]_{\langle \omega'_\alpha, \omega_\beta \rangle^{-1}} &&= \mathcal{F}_\beta \mathcal{F}_\alpha - \langle \omega'_\alpha, \omega_\beta \rangle^{-1} \mathcal{F}_\alpha \mathcal{F}_\beta.
\end{aligned}$$

Theorem (B06,BH08,C08,CHW23,HW09,HW10)

$\{\prod^{>}_{\alpha\in\Phi^+} \mathcal{E}_\alpha^{n_\alpha} \mid n_\alpha \in \mathbb{N}\}$ *and* $\{\prod^{<}_{\alpha\in\Phi^+} \mathcal{F}_\alpha^{n_\alpha} \mid n_\alpha \in \mathbb{N}\}$ *are convex PBW-type Lyndon bases of the algebra* U^+ *and* U^-*, respectively.*

[B06] Bai, X. (2006) *Two-parameter quantum groups of type E-series & restricted two-parameter quantum groups of type D*. Ph.D. Dissertation, East China Normal University.
[C08] Chen, R. (2008) *Restricted two-parameter quantum groups of type C*. Ph.D. Dissertation, East China Normal University.
[HW09] Hu, N., Wang, X. (2009) *Convex PBW-type Lyndon basis and restricted two-parameter quantum groups of type* G_2. Pacific J. Math. 241(2), 243–273.
[HW10] Hu, N., Wang, X. (2010) *Convex PBW-type Lyndon bases and restricted two-parameter quantum groups of type* B. J. Geom. Phys. 60(3), 430–453.

Cartan Part of Universal R-Matrix ($\mathcal{R}^0$)

Key Definitions

- $\dot{\ln} A$: Matrix with entries $(\dot{\ln} A)_{ij} = \ln(a_{ij}) = h\langle j, i\rangle - h'\langle i, j\rangle$
- Dual elements: $H_i^* = hh' \sum_{k=1}^{n} [(\dot{\ln} A)^{-1}]_{ik} H'_k$ (satisfy $\langle H_i^*, H_j\rangle = \delta_{ij}$)

Theorem (Cartan Part $\mathcal{R}^0$)

The universal R-matrix of Cartan part $U^0_{h,h'}(\mathfrak{g})$ is:

$$\mathcal{R}^0 = \exp\left(\sum_{i=1}^{n} H_i \otimes H_i^*\right) = \exp\left(hh' \sum_{i,j=1}^{n} [(\dot{\ln} A)^{-1}]_{ij} H_i \otimes H'_j\right)$$

Quasi-R-Matrix ($\mathcal{R}^+$)

Key lemma [Ro02, Cor.29] (Parallel result in costandard factorization)

Let $\Delta(\mathcal{E}_\beta) = \mathcal{E}_\beta \otimes 1 + \omega_\beta \otimes \mathcal{E}_\beta + \mathsf{Par}(\mathcal{E}_\beta)$, The summation $\mathsf{Par}(\mathcal{E}_\beta)$ has the factorizable property:

$$\mathsf{Par}(\mathcal{E}_\beta) = \sum (*)\ \mathcal{E}_{\alpha_{i_1}} \cdots \mathcal{E}_{\alpha_{i_k}} \omega_{\Sigma \alpha_{j_\iota}} \otimes \mathcal{E}_{\alpha_{j_1}} \cdots \mathcal{E}_{\alpha_{j_{k'}}},$$

where the summation is made on non-increasing products $\alpha_{i_1} \geqslant \cdots \geqslant \alpha_{i_k}$, $\alpha_{j_1} \geqslant \cdots \geqslant \alpha_{j_{k'}} > \beta > \alpha_{i_k}$, and $(*)$ are some scalars.

[Ro02] Rosso, M. (2002) *Lyndon bases and the multiplicative formula for R-matrices*. (preprint)

Theorem (Orthogonal Dual Bases of U^{+} and U^{-})

For $\mathcal{E}^{\boldsymbol{u}} = \prod^{\succ}_{\alpha\in\Phi^+} \mathcal{E}_\alpha^{u_\alpha}$ *and* $\mathcal{F}^{\boldsymbol{v}} = \prod^{\succ}_{\alpha\in\Phi^+} \mathcal{F}_\alpha^{v_\alpha}$*:*

$$\langle \mathcal{F}^{\boldsymbol{u}}, \mathcal{E}^{\boldsymbol{v}} \rangle = \delta_{\boldsymbol{u},\boldsymbol{v}} \prod^{\succ}_{\alpha\in\Phi^+} (u_\alpha)!_{\langle \omega'_\alpha, \omega_\alpha \rangle} \langle \mathcal{F}_\alpha, \mathcal{E}_\alpha \rangle^{u_\alpha}$$

where $(m)!_q = \prod_{k=1}^{m} \frac{1-q^k}{1-q}$ *(q-factorial)*

Values of the pairing $\langle \mathcal{F}_\alpha, \mathcal{E}_\alpha \rangle$

The the nonzero $\langle \mathcal{F}_\theta, \mathcal{E}_\theta \rangle$ can be determined by induction.

$$\langle \mathcal{F}_{\alpha_i}, \mathcal{E}_{\alpha_i} \rangle = (s_i - r_i)^{-1}$$

$$\begin{aligned}
\langle \mathcal{F}_\theta, \mathcal{E}_\theta \rangle &= \langle [\mathcal{F}_\psi, \mathcal{F}_\phi]_{\langle \omega'_\phi, \omega_\psi \rangle^{-1}} \ , \ \Delta(\mathcal{E}_\theta) \rangle \\
&= \langle \mathcal{F}_\psi \otimes \mathcal{F}_\phi - \langle \omega'_\phi, \omega_\psi \rangle^{-1} \mathcal{F}_\phi \otimes \mathcal{F}_\psi \ , \ \mathcal{E}_\theta \otimes 1 + \omega_\theta \otimes \mathcal{E}_\theta + \mathrm{Par}(\mathcal{E}_\theta) \rangle \\
&= (k^\theta_{\psi,\phi} - \langle \omega'_\phi, \omega_\psi \rangle^{-1} k^\theta_{\phi,\psi}) \cdot \langle \mathcal{F}_\phi, \mathcal{E}_\phi \rangle \langle \mathcal{F}_\psi, \mathcal{E}_\psi \rangle \ ,
\end{aligned}$$

$k^\theta_{\psi,\phi} = 0$ by Key Lemma.

The detailed value of $k^\theta_{\phi,\psi} \neq 0$ is determined either by applying the skew-derivation ∂_ψ to the root vectors $\mathcal{E}_\theta$, or by employing the detailed expressions of $\Delta(\mathcal{E}_\theta)$ established in the existing literature.

As introduced in [BW04,BGH07], for each simple root α_i, there is a skew-derivation $\partial_i : U^+_\gamma \to U^+_{\gamma-\alpha_i}$ for $\gamma \in Q^+$ defined inductively by

$$\partial_i(1) = 0, \quad \partial_i(e_j) = \delta_{ij}, \quad \partial_i(xx') = \langle \omega'_{\gamma'}, \omega_i \rangle \partial_i(x)x' + x\partial_i(x'),$$

for all $x \in U^+_\gamma$ and $x' \in U^+_{\gamma'}$. When applying it to the root vectors $\mathcal{E}_\theta$, one can verify directly that

Lemma (Computing partial operator)

For any non-simple root $\gamma = \alpha_{i_1 i_2 \cdots i_k i_{k+1}}$ whose corresponding path in the Lyndon tree is sequentially labeled by $i_1, i_2, \cdots, i_k, i_{k+1}$, the following identity holds for all $m = 1, \ldots, n$ that

$$\partial_m(\mathcal{E}_{i_1 \cdots i_k i_{k+1}}) = \delta_{m, i_{k+1}} (1 - \langle \omega'_{i_{k+1}}, \omega_{i_1 \cdots i_k} \rangle \langle \omega'_{i_1 \cdots i_k}, \omega_{i_{k+1}} \rangle)\ \mathcal{E}_{i_1 \cdots i_k} + \langle \omega'_{i_{k+1}}, \omega_m \rangle\ [\partial_m(\mathcal{E}_{i_1 \cdots i_k}), e_{i_{k+1}}]_{\langle \omega'_{k+1}, \omega_{i_1 \cdots i_k} \omega_m^{-1} \rangle}.$$

The Universal R-Matrix

From the theorem of orthogonal dual bases of U^+ and U^-, we have

Theorem (Quasi-R-Matrix $\mathcal{R}^+$)

$$\mathcal{R}^+ = \prod_{\alpha\in\Phi^+}^{\succ} \exp_{\langle\omega'_\alpha,\omega_\alpha\rangle}\left(\frac{\mathcal{E}_\alpha\otimes\mathcal{F}_\alpha}{\langle\mathcal{F}_\alpha,\mathcal{E}_\alpha\rangle}\right), \quad \textit{where} \quad \exp_q(x) = \sum_{m=0}^{\infty}\frac{x^m}{(m)!_q}$$

Main result: Universal R-Matrix

$$\begin{aligned}\mathcal{R} &= \mathcal{R}^+\mathcal{R}^0 \in U_{h,h'}(\mathfrak{g})\widehat{\otimes}U_{h,h'}(\mathfrak{g}) \\ &= \left(\prod_{\alpha\in\Phi^+}^{\succ} \exp_{\langle\omega'_\alpha,\omega_\alpha\rangle}\frac{\mathcal{E}_\alpha\otimes\mathcal{F}_\alpha}{\langle\mathcal{F}_\alpha,\mathcal{E}_\alpha\rangle}\right)\exp\left(hh'\sum_{i,j=1}^{n}\left[(\dot{\ln}A)^{-1}\right]_{ij}H_i\otimes H'_j\right),\end{aligned}$$

Application 1: Basic R-Matrix for Classical Types

Natural Representations of Classical Types

- Type A_n: $L(\varpi_1)$ of type 1 and $L(\varepsilon_1)$ of type $\theta : \alpha_i \mapsto (rs)^{\frac{1}{n+1}}$

$$V^\theta \otimes L(\varpi_1) \overset{\phi}{\cong} L(\varepsilon_1)$$

- Type $B_n/C_n/D_n$: $L(\varpi_1)$ (natural N-dimensional representation)

Basic R-Matrix Calculation: $R = (T_1 \otimes T_1)(\mathcal{R})$, where T_1 is the natural representation

Corollary

(Type A, $L(\varepsilon_1)$) *Recover the basic R-matrix of $U_{r,s}(\mathfrak{sl}_{N=n+1})$ in* [BW04]*:*

$$\sum_{i=1}^{n+1} E_{ii} \otimes E_{ii} + r \sum_{i<j} E_{ii} \otimes E_{jj} + s^{-1} \sum_{i>j} E_{ii} \otimes E_{jj} + (1 - rs^{-1}) \sum_{i<j} E_{ij} \otimes E_{ji};$$

(Type A, $L(\varpi_1)$) *Compute the basic R-matrix of $U_{r,s}(\mathfrak{sl}_{N=n+1})$:*

$$\sum_{i=1}^{n+1} E_{ii} \otimes E_{ii} + s^{-1} \sum_{k=1}^{n} \sum_{i=1}^{n+1} (rs)^{\frac{k}{n+1}} E_{\overline{i+k},\overline{i+k}} \otimes E_{ii} + (1 - rs^{-1}) \sum_{i<j} E_{ij} \otimes E_{ji}.$$

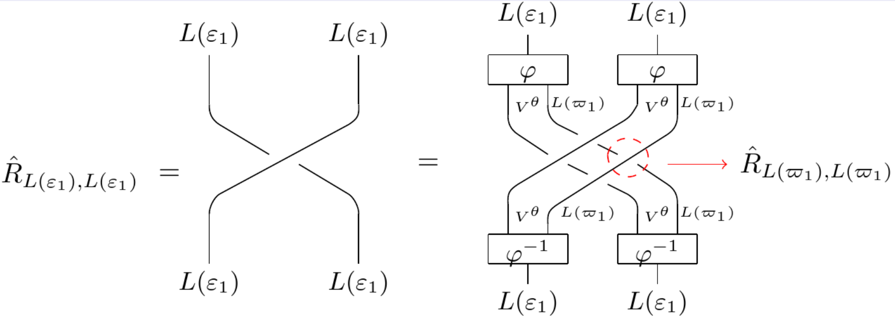
where $\overline{x}$ is the number in the range $\{1, 2, \cdots, n+1\}$ that is congruent to x modulo $n+1$.

[BW04] Benkart, G., Witherspoon, S. (2004) *Two-parameter quantum groups and Drinfel'd doubles*. Algebras Represent. Theory 7(3), 261–286.

Remark

Now, one could calculate the relationship between two basic R-matrices in the braided tensor category $\mathrm{Rep}(U_{r,s}(\mathfrak{g}))$:

$$c_{X,Y} = \widehat{R}_{X,Y} = P \circ \mathcal{R}|_{X \otimes Y}$$



Type BCD, $L(\varpi_1)$

Recover the basic R-matrices of $U_{\sqrt{r},\sqrt{s}}(\mathfrak{so}_{N=2n+1})$, $U_{r,s}(\mathfrak{sp}_{N=2n})$ and $U_{r,s}(\mathfrak{so}_{N=2n})$ in [HXZ24, ZHJ24, ZHX24]:

$$\begin{aligned}
&r^{-\frac{1}{2}}s^{\frac{1}{2}}\sum_{i\neq\frac{N+1}{2}}E_{ii}\otimes E_{ii}+r^{\frac{1}{2}}s^{-\frac{1}{2}}\sum_{i\neq\frac{N+1}{2}}E_{ii}\otimes E_{i'i'}+r^{-\frac{1}{2}}s^{-\frac{1}{2}}\sum_{(i,j)\in\mathrm{I}}E_{ii}\otimes E_{jj}+r^{\frac{1}{2}}s^{\frac{1}{2}}\sum_{(i,j)\in\mathrm{II}}E_{ii}\otimes E_{jj}\\
&\quad+E_{\frac{N+1}{2},\frac{N+1}{2}}\otimes E_{\frac{N+1}{2},\frac{N+1}{2}}+\sum_{i\neq\frac{N+1}{2}}E_{ii}\otimes E_{\frac{N+1}{2},\frac{N+1}{2}}+\sum_{i\neq\frac{N+1}{2}}E_{\frac{N+1}{2},\frac{N+1}{2}}\otimes E_{ii}\\
&\quad+(r^{-\frac{1}{2}}s^{\frac{1}{2}}-r^{\frac{1}{2}}s^{-\frac{1}{2}})\left(\sum_{i<j}E_{ij}\otimes E_{ji}-\sum_{i<j}(r^{-\frac{1}{2}}s^{\frac{1}{2}})^{\rho_j-\rho_i}\tau_i\tau_jE_{ij}\otimes E_{i'j'}\right),
\end{aligned}$$

where the second line is omitted for $U_{r,s}(\mathfrak{sp}_{2n})$, $U_{r,s}(\mathfrak{so}_{2n})$. The subscript sets I, II are shown in Figure. The notation $i' := N+1-i$,

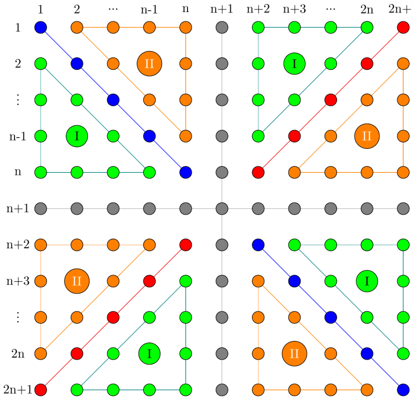
[HXZ24] Hu, N., Xu, X., Zhuang, R. (2024) *RLL-realization of two-parameter quantum affine algebra of type $B_n^{(1)}$ and regulated quantum Lyndon bases.* arXiv:2405.06587.

[ZHJ24] Zhong, X., Hu, N., Jing, N. (2026) *RLL-realization of two-parameter quantum affine algebra of type $C_n^{(1)}$*. J. Algebra Appl. (to appear).

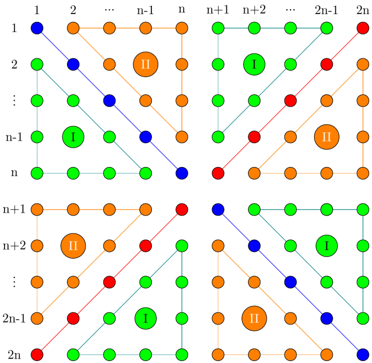
[ZHX24] Zhuang, R., Hu, N., Xu, X. (2024) *RLL-realization of two-parameter quantum affine algebra of type $D_n^{(1)}$*. Pacific J. Math. 329 (2), 357–395.

$$(\rho_1, \rho_2, \cdots, \rho_N) = \begin{cases} \left(n-\frac{1}{2}, n-\frac{3}{2}, \cdots, \frac{3}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{3}{2}, \cdots, -n+\frac{3}{2}, -n+\frac{1}{2}\right), & \mathfrak{g} = \mathfrak{so}_{2n+1}, \\ (n, n-1, \cdots, 2, 1, -1, -2, \cdots, -n+1, -n), & \mathfrak{g} = \mathfrak{sp}_{2n}, \\ (n-1, n-2, \cdots, 1, 0, 0, -1, \cdots, -n+2, -n+1), & \mathfrak{g} = \mathfrak{so}_{2n}, \end{cases}$$

$$\tau_i = \begin{cases} 1, & i \leqslant n \\ -1, & i > n \end{cases}, \quad \text{for } \mathfrak{g} = \mathfrak{sp}_{2n}, \qquad \tau_i \equiv 1, \quad \text{for } \mathfrak{g} = \mathfrak{so}_N.$$



Type B_n, so_{2n+1}



Type C_n, sp_{2n}; Type D_n, so_{2n}

Figure: Image of $\mathcal{R}^0$: Windmill symmetry in subscript sets I & II

Corollary (recently, Type G_2, $L(\varpi_1)$)

$$(T_1 \otimes T_1)(\mathcal{R}) = \sum_{i,j=1}^{7} T_{ij} E_{ii} \otimes E_{jj} + \sum_{i,j:\ i<j} \Big(X_{ij} E_{ij} \otimes E_{ji} + Y_{ij} E_{ij} \otimes E_{\bar{i}\bar{j}} \Big)$$

$$+ \sum_{(n,k,l)} \Big(M^{\mathrm{I}}_{n,k,l} E_{1,n} \otimes E_{k,l} + N^{\mathrm{I}}_{n,k,l} E_{1,n} \otimes E_{\bar{l},\bar{k}}$$

$$+ M^{\mathrm{II}}_{n,k,l} E_{l,k} \otimes E_{n,1} + N^{\mathrm{II}}_{n,k,l} E_{l,k} \otimes E_{\bar{1},\bar{n}}$$

$$+ M^{\mathrm{III}}_{n,k,l} E_{\bar{k},\bar{l}} \otimes E_{n,1} + N^{\mathrm{III}}_{n,k,l} E_{\bar{k},\bar{l}} \otimes E_{\bar{1},\bar{n}}$$

$$+ M^{\mathrm{IV}}_{n,k,l} E_{\bar{n},\bar{1}} \otimes E_{\bar{l},\bar{k}} + N^{\mathrm{IV}}_{n,k,l} E_{\bar{n},\bar{1}} \otimes E_{k,l} \Big)$$

$$T = \begin{pmatrix}
\frac{s}{r} & rs^2 & \frac{1}{r^2 s} & 1 & r^2 s & \frac{1}{rs^2} & \frac{r}{s} \\
\frac{1}{r^2 s} & \frac{s}{r} & \frac{1}{rs^2} & 1 & rs^2 & \frac{r}{s} & r^2 s \\
rs^2 & r^2 s & \frac{s}{r} & 1 & \frac{r}{s} & \frac{1}{r^2 s} & \frac{1}{rs^2} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\frac{1}{rs^2} & \frac{1}{r^2 s} & \frac{r}{s} & 1 & \frac{s}{r} & r^2 s & rs^2 \\
r^2 s & \frac{r}{s} & rs^2 & 1 & \frac{1}{rs^2} & \frac{s}{r} & \frac{1}{r^2 s} \\
\frac{r}{s} & \frac{1}{rs^2} & r^2 s & 1 & \frac{1}{r^2 s} & rs^2 & \frac{s}{r}
\end{pmatrix}$$

$$X=\begin{pmatrix} 0 & \frac{-r+s}{r} & \frac{-r+s}{r} & \frac{-r^4+r^3s-rs^3+s^4}{rs^3} & \frac{-r^4+r^2s^2-rs^3+s^4}{rs^3} & \frac{-r^4+r^2s^2-rs^3+s^4}{rs^3} & \frac{(r-s)^2(r+s)(r^2+s^2)^2}{rs^6} \\ 0 & 0 & \frac{-r^3+s^3}{rs^2} & \frac{-r^2+s^2}{rs} & \frac{-r+s}{r} & \frac{r^6-r^5s-rs^5+s^6}{rs^5} & \frac{-r^4+r^2s^2-rs^3+s^4}{rs^3} \\ 0 & 0 & 0 & \frac{-r^2+s^2}{rs} & \frac{(r-s)^2(r+s)}{rs^2} & \frac{-r+s}{r} & \frac{-r^4+r^2s^2-rs^3+s^4}{rs^3} \\ 0 & 0 & 0 & 0 & \frac{-r^2+s^2}{rs} & \frac{-r^2+s^2}{rs} & \frac{-r^4+r^3s-rs^3+s^4}{rs^3} \\ 0 & 0 & 0 & 0 & 0 & \frac{-r^3+s^3}{rs^2} & \frac{-r+s}{r} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-r+s}{r} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Y=\begin{pmatrix} 0 & \frac{-r+s}{rs^3} & \frac{r^2(r-s)}{s^3} & \frac{-r^4+r^3s-rs^3+s^4}{rs^3} & \frac{(r-s)(r^3+rs^2+s^3)}{s^4} & -\frac{(r-s)(r^3+rs^2+s^3)}{s^7} & 0 \\ 0 & 0 & -\frac{r(r^3-s^3)}{s} & \frac{r^4-r^2s^2}{s} & -\frac{r^4(r-s)}{s^2} & 0 & -\frac{r^3(r-s)(r^3+rs^2+s^3)}{s^4} \\ 0 & 0 & 0 & \frac{-r^2+s^2}{rs} & 0 & -\frac{r(r-s)}{s^5} & \frac{(r-s)(r^3+rs^2+s^3)}{s^4} \\ 0 & 0 & 0 & 0 & \frac{-r^2+s^2}{s^2} & \frac{(r-s)(r+s)}{s^5} & \frac{-r^4+r^3s-rs^3+s^4}{s^4} \\ 0 & 0 & 0 & 0 & 0 & \frac{-r^3+s^3}{r^2s^4} & \frac{r^2(r-s)}{s^3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -r^2(r-s) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M^{\square}=\begin{pmatrix} \mathrm{I}: & \left\{\frac{-r^2+s^2}{r^2s^3},\ \frac{r^2-s^2}{s^2},\ \frac{r(r-s)(r+s)}{s^3}\right\} \\ \mathrm{II}: & \left\{-rs(r-s),\ \frac{r(r-s)}{s^2},\ \frac{r(r-s)}{s^2}\right\} \\ \mathrm{III}: & \left\{\frac{-r+s}{s},\ \frac{r(r-s)}{s^2},\ \frac{-r+s}{s}\right\} \\ \mathrm{IV}: & \left\{\frac{-r^2+s^2}{rs},\ \frac{(r-s)(r+s)}{rs^4},\ \frac{r^4-r^2s^2}{s}\right\} \end{pmatrix},\quad N^{\square}=\begin{pmatrix} \mathrm{I}: & \left\{\frac{-(r^2-s^2)}{rs},\ \frac{r^2-s^2}{s^2},\ \frac{-(r^2-s^2)}{rs}\right\} \\ \mathrm{II}: & \left\{\frac{-(r-s)}{r^2s^2},\ \frac{r-s}{rs^3},\ -rs(r-s)\right\} \\ \mathrm{III}: & \left\{\frac{-(r-s)}{s},\ r^2(r-s),\ \frac{r-s}{s^4}\right\} \\ \mathrm{IV}: & \left\{-r(r^2-s^2),\ \frac{r^4-r^2s^2}{s},\ \frac{-(r^2-s^2)}{r^2s^3}\right\} \end{pmatrix}.$$

$(n,k,l)=(2,4,3),(4,6,3),(3,4,2),\ \bar{n}:=8-n$

The weight module $V = L(\varpi_1)$ has the following property

$$
\begin{array}{rccccccc}
V \otimes V \cong & S^0(V \otimes V) & \oplus & V & \oplus & \Lambda(V \otimes V) & \oplus & S'(V \otimes V) \\
\cong & L(0) & \oplus & L(\varpi_1) & \oplus & L(\varpi_2) & \oplus & L(2\varpi_1), \\
\mathsf{dim}: & 1 & & 7 & & 14 & & 27 \\
\mathsf{Spec}(\widehat{R}): & \dfrac{r^6}{s^6} & & -\dfrac{r^3}{s^3} & & -1 & & \dfrac{s}{r}
\end{array}
$$

with associated highest weight vectors:

$$
\begin{aligned}
L(0): \quad & \mathfrak{v}_0 = v_{1,7} - r^2 s\, v_{2,6} + \frac{r^2}{s^2}\, v_{3,5} - \frac{r^3}{s^3}\, v_{4,4} + \frac{r^3}{s^3}\, v_{5,3} - \frac{r^6}{s^3}\, v_{6,2} + \frac{r^5}{s^5}\, v_{7,1}, \\
L(\varpi_1): \quad & \mathfrak{v}_{\varpi_1} = v_{1,4} - r^2 s\, v_{2,3} + \frac{r^2}{s^2}\, v_{3,2} - \frac{r^3}{s^3}\, v_{4,1}, \\
L(\varpi_2): \quad & \mathfrak{v}_{\varpi_2} = v_{1,2} - r^2 s\, v_{2,1}, \\
L(2\varpi_1): \quad & \mathfrak{v}_{2\varpi_1} = v_{1,1}.
\end{aligned}
$$

The minimal polynomial of $\widehat{R}$ is

$$
\left(t - r^6 s^{-6}\right)\left(t + r^3 s^{-3}\right)(t+1)\left(t - r^{-1} s\right).
$$

G_2-FRT [HW26']

The unital associative algebra $U(R)$ is generated by ℓ_{ij}^{+}, ℓ_{ji}^{-}, $1 \leqslant i < j \leqslant 7$, and invertible elements ℓ_{ii}^{+}, ℓ_{ii}^{-}, $1 \leqslant i \leqslant 7$, with the defining relation

$$RL_1^{\pm}L_2^{\pm} = L_2^{\pm}L_1^{\pm}R, \quad RL_1^{+}L_2^{-} = L_2^{-}L_1^{+}R,$$

$$L^{\pm}C\,L^{\pm t}C^{-1} = I,$$

$$DL^{\pm} = L_1^{\pm t}L_2^{\pm t}D,$$

where $L^{\pm} = (\ell_{ij}^{\pm})$ with $\ell_{ij}^{+} = \ell_{ji}^{-} = 0$ when $1 \leqslant j < i \leqslant 7$, matrices $L_1^{\pm} = L^{\pm} \otimes 1$, $L_2^{\pm} = 1 \otimes L^{\pm}$, and matrices C and D defined by the Clebsch–Gordan coefficients from the injections $L(0) \hookrightarrow V \otimes V$ and $V = L(\varpi_1) \hookrightarrow V \otimes V$

[HW26'] Hu, N., Wang, H. (2026) *On Two-parameter Quantum Groups: FRT-Formalism and RLL-Realization of Type* G_2. (work in progress)

Application 2: Central Elements

Theorem [HW26] Harish-Chandra theorem for $U_{r,s}(\mathfrak{g})$

- The HC map $\xi = \gamma^{-\rho}\pi : \ Z(U_{r,s}(\mathfrak{g})) \longrightarrow U^0$ is injective.
- For $\lambda \in P^+ \cap Q$, $\exists! z_\lambda \in Z(U)$, s.t. $\langle z_\lambda | - \rangle = \mathrm{tr}_{L(\lambda)}(- \circ \Theta)$, and

$$z_\lambda = \sum_{\tau \leqslant \lambda} \sum_{\mu \in Q^+} \sum_{i,j} (rs^{-1})^{-(\rho,\tau+\mu)} \langle \omega'_\mu, \omega_{\tau+\mu} \rangle tr(v_j^\mu u_i^\mu \circ P_\tau) \ v_i^\mu \omega'_\tau \omega_{\tau+\mu}^{-1} u_j^\mu .$$

- When rank n is even, $\xi : \ Z(U_{r,s}(\mathfrak{g})) \cong (U_\flat^0)^W$.
  When rank n is odd, $\mathrm{Im}(\xi) \supseteq (U_\flat^0)^W \otimes \mathbb{K}[z_*, z_*^{-1}]$.

[HW26] Hu, N., Wang, H. (2026) *Harish-Chandra Theorem for Two-parameter Quantum Groups*. Forum Math. 38 (1), 193–214.

The explicit generators and defining relations for $Z(U)$ are subsequently given in [CHW25], analogous to those for $U_q(\mathfrak{g})$ presented in [LXZ16].

Theorem [HW26] The weight lattice type $\breve{U}_{r,s}(\mathfrak{g})$

- When n is even, the centre $Z(\breve{U}_{r,s}) \cong (\breve{U}^0_\flat)^W \cong K(U_{r,s}) \cong K(U_q)$, and

$$Z(\breve{U}_{r,s}) = \mathbb{K}[z_{\varpi_1}, \cdots, z_{\varpi_n}].$$

  When n is odd, the centre $Z(\breve{U}_{r,s}) \supseteq \mathbb{K}[z_{\varpi_1}, \cdots, z_{\varpi_n}] \otimes \mathbb{K}[z_*^{\frac{1}{\ell}}, z_*^{-\frac{1}{\ell}}]$, where $\ell = 2$, except $\ell = 4$ for D_{2k+1}.

[LXZ16] Li, L., Xia, L., Zhang, Y. (2016) *On the center of the quantized enveloping algebra of a simple Lie algebra*. arXiv:1607.00802.

[CHW25] Chen, K., Hu, N., Wang, H. (2025) *Harish-Chandra Theorem for the Multi-Parameter Quantum Groups of Okado-Yamane Type*. arXiv:2505.18599

Application 2: Γ operator

Central elements can be constructed by taking the quantum trace of certain operators $\Gamma \in U_{r,s}(\mathfrak{g}) \otimes \mathrm{End}(L(\lambda))$, analogous to the construction in $U_q(\mathfrak{g})$ [ZGB91]. It requires the map $\Theta : M \to M$ defined by

$$m \mapsto (rs^{-1})^{-(\rho,\lambda)} m, \quad \forall m \in M_\lambda,\ \lambda \in P.$$

Theorem [HW26]

Let $\lambda \in P^+$ and $\zeta : U_{r,s}(\mathfrak{g}) \to \mathrm{End}(L(\lambda))$ be the weight representation. If there is an operator $\Gamma \in U_{r,s}(\mathfrak{g}) \otimes \mathrm{End}(L(\lambda))$ such that

$$\Gamma \circ (id \otimes \zeta)\Delta(x) = (id \otimes \zeta)\Delta(x) \circ \Gamma, \quad \forall\, x \in U_{r,s}(\mathfrak{g}),$$

then the element $c_\lambda = \mathrm{tr}_2(\Gamma(1 \otimes \Theta)) \in Z(U_{r,s}(\mathfrak{g}))$.

Write $\Gamma\Delta(x) = \Delta(x)\,\Gamma$ for short. Now we construct such an operator.

In $U_{h,h'}(\mathfrak{g})$ we have $\Delta^{\mathrm{op}}(x)\mathcal{R} = \mathcal{R}\Delta(x)$. Since $\mathcal{R} = \mathcal{R}^{+}\mathcal{R}^{0}$, let $\Psi(\cdot) = \mathcal{R}^{0}(\cdot)(\mathcal{R}^{0})^{-1}$ be an automorphism of $U_{h,h'}(\mathfrak{g})^{\widehat{\otimes}2}$. In fact, this automorphism can be restricted in $U_{r,s}(\mathfrak{g})^{\otimes 2}$.

Lemma

(1) *Let* $x_\alpha \in U_\alpha,\ y_\beta \in U_\beta,\ \alpha,\beta \in Q$ *and* $\omega_0 = \omega'_0 = 1$, *then*

$$\Psi(x_\alpha \otimes y_\beta) = \mathcal{R}^{0}(x_\alpha \otimes y_\beta)(\mathcal{R}^{0})^{-1} = x_\alpha\omega_\beta^{-1} \otimes \omega'_\alpha y_\beta.$$

(2) $\Psi|_{U_{r,s}(\mathfrak{g})^{\otimes 2}}$ *is an automorphism of* $U_{r,s}(\mathfrak{g})^{\otimes 2}$.

(3) *For convenience, this restriction is again denoted by* Ψ. *Then we have*

$$\Delta^{\mathrm{op}}(x)\mathcal{R}^{+} = \mathcal{R}^{+}\Psi(\Delta(x)),\ \forall x \in U_{r,s}(\mathfrak{g}).$$

To the target $\Gamma\Delta(x) = \Delta(x)\,\Gamma$, one needs another operators to cancel the effect of $^{\mathrm{op}}$ and Ψ.

Set operators: $\mathcal{K}_\lambda = \sum_{\mu\in Q} \omega_\mu^{-1} \otimes P_\mu$, $\mathcal{K}'_\lambda = \sum_{\mu\in Q} \omega'_\mu \otimes P_\mu$, where P_μ is the μ-weight space projector.

Theorem

For all weight $\lambda \in P^+ \cap Q$ and all $x \in U_{r,s}(\mathfrak{g})$, we have the following relations

$$\begin{aligned}
&(1)\quad \mathcal{K}_\lambda \Delta(x) = \Psi(\Delta(x))\mathcal{K}_\lambda, \qquad && \mathcal{K}'_\lambda \Delta^{\mathrm{op}}(x) = (\Psi\Delta(x))^{\mathrm{op}}\mathcal{K}'_\lambda;\\
&(2)\quad \mathcal{R}^+\mathcal{K}_\lambda \Delta(x) = \Delta^{\mathrm{op}}(x)\mathcal{R}^+\mathcal{K}_\lambda, \qquad && (\mathcal{R}^+)^{\mathrm{op}}\mathcal{K}'_\lambda \Delta^{\mathrm{op}}(x) = \Delta(x)(\mathcal{R}^+)^{\mathrm{op}}\mathcal{K}'_\lambda;\\
&(3)\quad \Gamma\Delta(x) = \Delta(x)\Gamma, \quad \textit{where}\ \ \Gamma := (\mathcal{R}^+)^{\mathrm{op}}\mathcal{K}'_\lambda\mathcal{R}^+\mathcal{K}_\lambda.
\end{aligned}$$

Theorem (Central Elements c_λ, $\lambda \in P^+ \cap Q$)

Let $\Gamma = (\mathcal{R}^+)^{\mathrm{op}} \mathcal{K}'_\lambda \mathcal{R}^+ \mathcal{K}_\lambda$.
The element $c_\lambda = \mathrm{Tr}_2\left(\Gamma(1 \otimes \Theta)\right)$ *belongs to the center* $Z(U_{r,s}(\mathfrak{g}))$, *and*

$$c_\lambda = \sum_{(\eta;\gamma,a,b)\in\mathcal{I}} \frac{\langle \omega'_\gamma, \omega_\eta \rangle \,\mathrm{Tr}(\mathcal{E}^{\langle a\rangle} \mathcal{F}^{\langle b\rangle} P_\eta)}{(rs^{-1})^{(\rho,\eta)} \mathcal{P}^{\langle a\rangle} \mathcal{P}^{\langle b\rangle}} \cdot \mathcal{F}^{\langle a\rangle} \omega'_{\eta-\gamma} \omega_\eta^{-1} \mathcal{E}^{\langle b\rangle},$$

where $\mathcal{P}^{\langle u\rangle} = \langle \mathcal{F}^{\langle u\rangle}, \mathcal{E}^{\langle u\rangle} \rangle$ *and the index set*

$$\mathcal{I} = \left\{(\eta;\gamma,a,b) \mid \gamma \in Q^+;\ \eta, \eta-\gamma \in \mathrm{Wt}(L(\lambda));\ a, b \in \mathcal{I}_n,\ |a| = |b| = \gamma\right\}.$$

In fact, the element $c_\lambda = z_\lambda$, which $\langle z_\lambda \mid - \rangle = \mathrm{tr}_{L(\lambda)}(- \circ \Theta)$.

While the proof for the Harish-Chandra theorem [HW26] relies solely on the existence and formal notation of the dual basis, we now supplement it by giving explicit expressions for all $\mathcal{E}_\alpha$, $\mathcal{F}_\alpha$ and $\mathcal{P}_\alpha$.

This method could be naturally extended to the two-parameter quantum group $\breve{U}_{r,s}(\mathfrak{g})$ of weight lattice type, whose Cartan part is generated by $\{\omega_{\varpi_i}^{\pm 1},\ \omega'^{\pm 1}_{\varpi_i}\}_{i=1}^{n}$.

Corollary

For each $\lambda \in P^+$, we define

$$\breve{\mathcal{K}}_\lambda = \sum_{\mu \in P} \omega_\mu^{-1} \otimes P_\mu, \qquad \breve{\mathcal{K}}'_\lambda = \sum_{\mu \in P} \omega'_\mu \otimes P_\mu$$

on $\breve{U}_{r,s} \otimes \mathrm{End}(L(\lambda))$. Then one could get

$$\breve{\Gamma} := (\mathcal{R}^+)^{\mathrm{op}} \breve{\mathcal{K}}'_\lambda \mathcal{R}^+ \breve{\mathcal{K}}_\lambda, \text{ and } c_\lambda = \mathrm{Tr}_2 \left(\breve{\Gamma}(1 \otimes \Theta) \right) \in Z(\breve{U}_{r,s}(\mathfrak{g})).$$

Application 3: two-paramater classical r-matrices

The classical r-matrix for any simple Lie algebra $\mathfrak{g}$ can be extracted from the universal R-matrix.
In $\mathfrak{sl}_2$ case we have:

$$\mathcal{R} = \exp_{rs^{-1}}\left((s-r)\, e \otimes f\right) \exp\left(\frac{hh'}{h-h'} H \otimes H'\right)$$

Take the classical limit by letting $\hbar \to 0$ when $h = \epsilon_1 \hbar$ and $h' = \epsilon_2 \hbar$:

$$r_{\mathrm{cl}} = (\epsilon_2 - \epsilon_1) e \otimes f + \frac{\epsilon_1 \epsilon_2}{\epsilon_1 - \epsilon_2} H \otimes H'$$

$$\delta(x) := [\Delta(x), r_{\mathrm{cl}}]$$

$$\delta(H) = \delta(H') = 0,\ \delta(e) = -\epsilon_1 e \wedge H,\ \delta(f) = \epsilon_2 f \wedge H'$$

Particularly, when $(\epsilon_1, \epsilon_2) = (1, -1)$, it recovers

$$2e \otimes f + \frac{1}{2} H \otimes H'$$

Further Directions

- Two-parameter quantum group of type G_2: FRT–Chevalley and RLL–Drinfeld isomorphism, and applications to G_2-spider invariants
- Applications to RT/TV invariants
- In root of unity settings (exotic small[HX26]), non-semisimple

[HX26] Hu, N., Xu, X. (2026) *Novel isoclasses of one-parameter exotic small quantum groups originating from a two-parameter framework*. Bull. Sci. Math. 206, 103738.

Thank you for your attention!