Homological Spectra of Monoidal Triangulated Categories

Milen Yakimov
Northeastern University and
International Center for Mathematical Sciences
Bulgarian Academy of Sciences

jt work with Daniel Nakano (Univ Georgia) and Kent Vashaw (UCLA)

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Setting

Monoidal Triangulated Category MΔC $(\mathbf{K}, \otimes, \mathbf{1})$:

1. A triangulated category $\mathbf{K}$, $\mathbb{k}$-linear over a field $\mathbb{k} = \overline{\mathbb{k}}$, any char $\mathbb{k}$,
2. $\otimes : \mathbf{K} \times \mathbf{K} \to \mathbf{K}$, biexact and associative,
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Initially, interest in Symmetric M∆Cs came from Stable Homotopy Theory and Algebraic Geometry.
E.g. $X =$ a scheme $\rightsquigarrow$ the symmetric M∆C

$$(\mathrm{D}^{\mathrm{perf}}(X), \otimes^{L}_{\mathcal{O}_X})$$

Hopf algebras and Finite Tensor Categories

H= a fin dim Hopf algebra over $\mathbb{k}$:

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$\rightsquigarrow$ Happel's stable category with induced tensor product:

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[Etingof–Ostrik] A finite tensor category is an abelian $\Bbbk$-linear monoidal category $(\mathbf{T}, \otimes, \mathbf{1})$ such that

- finitely many simples, every object has finite length and $\dim_{\Bbbk} \mathrm{Hom}_{\mathbf{T}}(A, B) < \infty$;
- there are enough projectives;
- $\mathbf{T}$ is rigid, i.e., A^* and ${}^*A$ exist with evaluation/coeval maps;
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- $\mathrm{End}_{\mathbf{T}}(\mathbf{1}) \cong \mathbb{k}$.

Consequences: $\mathbf{T}$ is Frobenius $\rightsquigarrow$ the stable category $(\underline{\mathbf{T}}, \otimes)$ an MΔC.

Thick ideals

Problem

Classify all thick $\otimes$ ideals of an MΔC, $\mathbf{K}$.

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Approach of Balmer for symmetric MΔCs. Define a prime $\otimes$ ideal $\mathbf{P}$ of $\mathbf{K}$ to be a proper thick $\otimes$ ideal such that

$$A \otimes B \in \mathbf{P} \quad \Rightarrow \quad A \in \mathbf{P} \text{ or } B \in \mathbf{P}.$$

The Balmer spectrum [2005] $\operatorname{Spc}\mathbf{K}$ consists of prime $\otimes$ ideals with the topology generated by

$$V(A) := \{\mathbf{P} \in \mathbf{K} \mid A \notin \mathbf{P}\} \quad \text{for} \quad A \in \mathbf{P}.$$

Commutative tensor triangular geometry

Theorem [Balmer 2005]

For a symmetric MΔC, $\mathbf{K}$ there is an order preserving bijection between

1. radical thick $\otimes$ ideals of $\mathbf{K}$ and
2. the collection of Thomason subsets of $\operatorname{Spc} \mathbf{K}$ (unions of closed subsets whose complements are quasi-compact).

Noncommutative Balmer spectrum

Definition. [Buan–Krause–Solberg 2007] A prime $\otimes$ ideal of an (arbitrary) MΔC, $\mathbf{K}$ is a proper thick $\otimes$ ideal $\mathbf{P}$ such that

$$\mathbf{I} \otimes \mathbf{J} \subseteq \mathbf{P} \quad \Rightarrow \mathbf{I} \subseteq \mathbf{P} \text{ or } \mathbf{J} \subseteq \mathbf{P}$$

for all thick $\otimes$ ideals $\mathbf{I}$ and $\mathbf{J}$. Equivalent to

$$A \otimes M \otimes B \in \mathbf{P}, \ \forall M \in \mathbf{K} \quad \Rightarrow \quad A \in \mathbf{P} \text{ or } B \in \mathbf{P}.$$

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The noncommutative Balmer spectrum $\operatorname{Spc} \mathbf{K}$ = the collection of prime ideals with topology generated by

$$V(A) := \{\mathbf{P} \in \operatorname{Spc} \mathbf{K} : A \notin \mathbf{P}\}, \quad A \in \mathbf{K}.$$

Classification of thick ideals

Theorem [Nakano–Vashaw–Y 2023]

Assume that $\mathbf{K}$ is a rigid MΔC generated by a single object as a thick subcategory. Then there is a bijection between

1. the thick $\otimes$ ideals of $\mathbf{K}$ and
2. the collection of Thomason subsets of $\operatorname{Spc}\mathbf{K}$.

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The Blamer support

$$V : \mathbf{K} \to \mathcal{X}_{cl}(\operatorname{Spc}\mathbf{K}), \quad V(A) := \{\mathbf{P} \in \operatorname{Spc}\mathbf{K} : A \notin \mathbf{P}\},$$

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satisfies the noncommutative tensor product property

$$\bigcup_{X \in \mathbf{K}} V(M \otimes X \otimes N) = V(M) \cap V(N), \quad \forall M, N \in \mathbf{K}$$

and is universal terminal object for such support maps.

The cohomological support

Quillen (1971), Alperin-Evens (1981), Avrunin-Scott (1982), Carlson (1983), Benson-Carlson-Rickard (1996), ...

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The cohomology ring of an MΔC $\mathbf{K}$ is the graded commutative ring

$$R^{\bullet}_{\mathbf{K}} := \bigoplus_{k \geq 0} \mathrm{Hom}_{\mathbf{K}}(\mathbf{1}, \Sigma^k \mathbf{1})$$

with product

$$f \cdot g = (\Sigma^j f) g = f \otimes g \quad \text{for} \quad f \in \mathrm{Hom}_{\mathbf{K}}(A, \Sigma^k B), g \in \mathrm{Hom}_{\mathbf{K}}(A, \Sigma^j B).$$

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Then

$$\mathrm{Hom}^\bullet_{\mathbf{K}}(A, B) := \bigoplus_{k\geq 0} \mathrm{Hom}_{\mathbf{K}}(A, \Sigma^k B)$$

is an $R^\bullet_{\mathbf{K}}$-bimodule because $\Sigma^k \mathbf{1} \otimes A \cong \Sigma^k A$. The cohomological support

$$W : \mathbf{K} \to \mathcal{X}_{cl}(\mathrm{Proj}\, R^\bullet_{\mathbf{K}}) \quad \text{is} \quad W(A) := \{\mathfrak{p} \in \mathrm{Proj}\, R^\bullet_{\mathbf{K}} : \mathrm{Ann}(\mathrm{End}^\bullet_{\mathbf{K}}(A)) \subseteq \mathfrak{p}\}.$$

The comparison problem

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Describe the relationship between $\operatorname{Spc} \mathbf{K}$ and $\operatorname{Proj} R^{\bullet}_{\mathbf{K}}$ and the two support maps (Balmer and cohomological).

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Describe the relationship between Spc $\mathbf{K}$ and Proj $R^{\bullet}_{\mathbf{K}}$ and the two support maps (Balmer and cohomological).

Theorem [Benson-Carlson-Rickard, Friedlander-Pevtsova]

For every finite group scheme G,

$$\operatorname{Spc}(\underline{\operatorname{mod}}(\Bbbk G)) \cong \operatorname{Proj} R^{\bullet}_{\underline{\operatorname{mod}}(\Bbbk G)}$$

and the Balmer and cohomological supports coincide under this identification.

Based on the tensor product property for the cohomological support, in turn based on rank support [Carlson], π-support [Friedlander-Pevtsova].

The categorical center of the cohomology ring

When **T** is not braided, there are known cases where the Balmer and cohomological support are not homeomorphic [Benson-Witherspoon 2014, Plavnik-Witherspoon].

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Definition [Nakano-Vashaw-Y 2021]

The categorical center $C^\bullet_{\mathbf{K}}$ of the cohomology ring $R^\bullet_{\mathbf{K}}$ of $\mathbf{K}$ is the subalgebra spanned by all $g \in \mathrm{Hom}_{\mathbf{K}}(\mathbf{1}, \Sigma^n \mathbf{1})$ such that the diagram

$$\begin{array}{ccccc}
\mathbf{1} \otimes M & \xrightarrow{\cong} & M & \xleftarrow[\cong]{} & M \otimes \mathbf{1} \\
\downarrow{\scriptstyle g \otimes \mathrm{id}_M} & & & & \downarrow{\scriptstyle \mathrm{id}_M \otimes g} \\
\Sigma^n \mathbf{1} \otimes M & \xrightarrow{\cong} & \Sigma^n M & \xleftarrow[\cong]{} & M \otimes \Sigma^n \mathbf{1}
\end{array}$$

commutes for a collection of objects M that generates $\mathbf{K}$.

Relations to other Centers

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In general, the categorical center is related to the following two previously considered centers:

$$R^\bullet_{\underline{\mathbf{Z}(\mathbf{T})}} \longrightarrow C^\bullet_{\underline{\mathbf{T}}} \hookrightarrow R^\bullet_{\underline{\mathbf{T}}} \xhookrightarrow{\mathcal{R},\mathcal{L}} Z^\bullet(\underline{\mathbf{T}})$$

1. The Drinfeld center (area of monoidal categories),
2. The graded center (area of triangulated categories).

The Drinfeld center

For a finite tensor category $\mathbf{T}$, one forms its Drinfeld center $\mathbf{Z}(\mathbf{T})$, which is a braided FTC.

- Objects: pairs (A, γ), $A \in \mathbf{T}$ and a natural isomorphism

$$\gamma_X : X \otimes A \xrightarrow{\cong} A \otimes X, \quad X \in \mathbf{T}$$

  called a half-braiding, satisfying usual braiding type axioms.
- Morphisms: $\mathrm{Hom}_{\mathbf{Z}(\mathbf{T})}(A, B) \subset \mathrm{Hom}_{\mathbf{T}}(A, B)$ commuting with half-braiding.

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MΔCs: The forgetful functor $\mathbf{Z}(\mathbf{T}) \to \mathbf{T}$ descends to $\underline{\mathbf{Z}(\mathbf{T})} \to \underline{\mathbf{T}}$, giving rise to the homomorphism of cohomology rings

$$R^{\bullet}_{\underline{\mathbf{Z}(\mathbf{T})}} \to R^{\bullet}_{\underline{\mathbf{T}}} \quad \text{image in} \quad C^{\bullet}_{\underline{\mathbf{T}}}$$

The Graded Center

The graded center $Z^\bullet(\mathbf{K})$ of $\mathbf{K}$ is a graded commutative algebra with degree n component consisting of natural transformations

$$\eta : \mathrm{id}_{\mathbf{K}} \to \Sigma^n \quad \text{such that} \quad \eta\Sigma = (-1)^n \Sigma\eta.$$

Two injective homomorphisms $\mathcal{L}, \mathcal{R} : R^\bullet_{\mathbf{K}} \hookrightarrow Z^\bullet(\mathbf{K})$, which send $g \in \mathrm{Hom}_{\mathbf{K}}(\mathbf{1}, \Sigma^n \mathbf{1})$ to

$$M \xrightarrow{\cong} \mathbf{1} \otimes M \xrightarrow{g \otimes \mathrm{id}_M} \Sigma^n \mathbf{1} \otimes M \xrightarrow{\cong} \Sigma^n M \quad \text{and}$$

$$M \xrightarrow{\cong} M \otimes \mathbf{1} \xrightarrow{\mathrm{id}_M \otimes g} M \otimes \Sigma^n \mathbf{1} \xrightarrow{\cong} \Sigma^n M,$$

respectively. The categorical center of the cohomology ring $R^\bullet_{\mathbf{K}}$ is the equalizer

$$C^\bullet_{\mathbf{K}} := \{g \in R^\bullet_{\mathbf{K}} \mid \mathcal{L}(g) = \mathcal{R}(g) \text{ on a set of generators } M \in \mathbf{K}\}.$$

A Cohomological Comparison Map

Conjecture [Etingof-Ostrik]. Every Finite Tensor Category $\mathbf{T}$ satisfies the finite generation conditions:

1. its cohomology ring $R^\bullet_{\underline{\mathbf{T}}}$ is a finitely generated algebra and
2. for all $M \in \mathbf{T}$, $\mathrm{End}^\bullet_{\underline{\mathbf{T}}}(M)$ is a finitely generated $R^\bullet_{\underline{\mathbf{T}}}$-module.

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Theorem [Nakano-Vashaw-Y 2021]

1. There is a well-defined, continuous map

$$\rho : \mathrm{Spc}\, \mathbf{K} \to \mathrm{Spec}^{\mathrm{h}}\, C^\bullet_{\mathbf{K}}$$

given by $\rho(\mathbf{P}) = \langle g \text{ homogeneous in } C^\bullet_{\mathbf{K}} : \mathrm{cone}(g) \notin \mathbf{P}\rangle$.

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2. If $\mathbf{K}$ satisfies weak finite generation (for all $M \in \mathbf{K}$, $\mathrm{End}^\bullet_{\mathbf{K}}(M)$ is a finitely generated $C^\bullet_{\mathbf{K}}$-module), then ρ is surjective to Proj.

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3. If, in addition, $\mathrm{Proj}\, C^{\bullet}_{\mathbf{K}}$ is a Noetherian topological space, and the central cohomological support satisfies the noncommutative tensor product property then ρ is a homeomorphism.

A conjecture

Example. The last part of the theorem applies to all Benson-Witherspoon examples.

Conjecture [Nakano-Vashaw-Y 2021]

For every Finite Tensor Category $\underline{\mathbf{T}}$, the map

$$\rho : \operatorname{Spc} \underline{\mathbf{T}} \to \operatorname{Proj} C^{\bullet}_{\underline{\mathbf{T}}}$$

is a homeomorphism.

The Freyd Envelope and the Yoneda Embedding

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The Day convolution product (right exact)

$$(\text{Mod-}\mathbf{K}, \otimes) \quad \hookleftarrow \quad (\text{mod-}\mathbf{K}, \otimes), \quad \text{M}\Delta\text{Cs}$$

and the Yoneda embedding $h : (\mathbf{K}, \otimes) \to (\text{mod-}\mathbf{K}, \otimes)$ is monoidal.

Homological primes and the homological spectrum

The inverse images under the Yoneda embedding $h : \mathbf{K} \to \text{mod-}\mathbf{K}$ define a continuous map

$$\widetilde{\phi} : (\text{Serre prime spectrum of mod-}\mathbf{K}) \to \operatorname{Spc} \mathbf{K} \ (\text{Balmer spectrum of } \mathbf{K}),$$

$$\widetilde{\phi}(\mathbf{S}) := h^{-1}(\mathbf{S}).$$

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Definition [Nakano–Vashaw–Y 2025]

1. The homological primes of an MΔC, $\mathbf{K}$ are the maximal elements (with respect to inclusion) in each fiber of $\widetilde{\phi}$.
2. The homological spectrum $\operatorname{Spc}^h(\mathbf{K})$ of $\mathbf{K}$ is the collection of homological primes with the topology generated by

$$V^h(A) := \{\mathbf{S} \in \operatorname{Spc}^h(\mathbf{K}) : h(A) \notin \mathbf{S}\}, \quad \forall A \in \mathbf{K}.$$

Symmetric $\mathbf{K}$: [Balmer 2017] used maximal Serre ideals of mod-$\mathbf{K}$.

The homological comparison map

The homological support map is defined by

$$V^h : \mathrm{Spc}\,\mathbf{K} \to \mathcal{X}_{cl}(\mathrm{Spc}^h(\mathbf{K})), \quad A \mapsto V^h(A).$$

It has the noncommutative tensor product property

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Theorem. [Nakano–Vashaw–Y 2025]

The restricted map (from $\widetilde{\phi}$)

$$\phi : \operatorname{Spc}^h \mathbf{K} \to \operatorname{Spc} \mathbf{K}$$

is a surjective continuous map.

The Nerves of Steel Conjecture

Nerves of Steel Conjecture 2017, 2025

For every monoidal triangulated category $\mathbf{K}$, the homological support map ϕ is a homeomorphism from the homological spectrum of $\mathbf{K}$ to the Blamer spectrum of $\mathbf{K}$.

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Theorem [Nakano–Vashaw–Y 2025]

If the nerves of steel conjecture holds for a monoidal triangulated category $\mathbf{K}$, $\operatorname{Spc}\mathbf{K}$ is Noetherian, and G is a (not necessarily finite) group acting on $\mathbf{K}$ by monoidal triangulated autoequivalences, then the nerves of steel conjecture holds for the crossed product category $\mathbf{K} \ltimes G$.

Proof uses results of Hongdi Huang and Kent Vashaw.

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Corollary. Nerves of steel conjecture holds for the Benson-Witherspoon Hopf algebras $(\Bbbk[L]\#\Bbbk G)^*$, where L and G and groups and L is finite.

Presented results in:

1. *Noncommutative tensor triangular geometry*, Amer. J. Math. **144** (2022), no. 6, 1681–1724.
2. *On the spectrum and support theory of a finite tensor category*, Math. Ann. **390** (2024), 205-254.
3. *A Chinese remainder theorem and Carlson's theorem for monoidal triangulated categories*, arXiv:2311.17883.
4. *The homological spectrum for monoidal triangulated categories*, arXiv:2506.19946.