Monoidal categories graded by 2-groups

Alexis Virelizier
University of Lille

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Topological motivation for graded structures

- Monoidal categories play a central role in quantum topology: they are used to define quantum invariants
- **Goal:** extend quantum invariants of 3-manifolds to invariants of 3-manifolds with extra structure
- Encode the extra structure with a homotopy class of maps to a target X (viewed as the classifying space of the structure)

Ex: $X = BG$ with G a group $\leftrightsquigarrow$ principal G-bundles
$X = B\mathcal{G}$ with $\mathcal{G}$ a 2-group $\leftrightsquigarrow$ principal $\mathcal{G}$-2-bundles

$\rightsquigarrow$ invariants of pairs $\big(M$ closed oriented 3-manifold, $h \in [M, X]\big)$

Example: cohomological invariants from $\theta \in H^3(X, \Bbbk^*)$

$$\tau^\theta(M, h) = \langle h^*(\theta), [M] \rangle \in \Bbbk$$

$h^*(\theta) \in H^3(M, \Bbbk^*)$ and $[M] \in H_3(M, \mathbb{Z})$ fundamental class of M

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A *n-type* is a connected CW-complex with $\pi_k = 0$ for $k > n$.

- **X is a 0-type:** $X \simeq \{\mathrm{pt}\}$

Turaev-Viro (1992), Barret-Westburry (1996)

$\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}(M)$

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Monoidal categories graded by a group G

A $\Bbbk$-linear monoidal category $\mathcal{C}$ is G-graded if

1. it decomposes as $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ (set $|X| = g$ for $X \in \mathcal{C}_g$)
   For homogenous objects: $\mathrm{Hom}_{\mathcal{C}}(X, Y) \neq 0 \Rightarrow |Y| = |X|$
2. $|X \otimes Y| = |X|\,|Y|$ and $|\mathbb{1}| = 1$

Ex: category of modules over a Hopf G-coalgebra $A = \{A_g\}_{g \in G}$

$\mathcal{C}_g = \mathrm{Mod}(A_g)$ $\otimes_{\mathcal{C}}$ induced by $\Delta_{g,h} \colon A_{gh} \to A_g \otimes_{\Bbbk} A_h$

A G-fusion category is a rigid G-graded $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ such that:

3. $\mathcal{C}$ is semisimple
4. each $\mathcal{C}_g$ has finitely many simple objects (up to iso)
5. the monoidal unit $\mathbb{1}$ simple

Ex:
- $\mathrm{Mod}_{fd}(A)$ with $A = \{A_g\}_{g \in G}$ semisimple of finite-type
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1. it decomposes as $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ (set $|X| = g$ for $X \in \mathcal{C}_g$)
   For homogenous objects: $\mathrm{Hom}_{\mathcal{C}}(X, Y) \neq 0 \Rightarrow |Y| = |X|$
2. $|X \otimes Y| = |X|\,|Y|$ and $|\mathbb{1}| = 1$

Ex: category of modules over a Hopf G-coalgebra $A = \{A_g\}_{g \in G}$

$\mathcal{C}_g = \mathrm{Mod}(A_g)$ $\otimes_{\mathcal{C}}$ induced by $\Delta_{g,h}\colon A_{gh} \to A_g \otimes_{\Bbbk} A_h$

A *G-fusion category* is a rigid G-graded $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ such that:

3. $\mathcal{C}$ is semisimple
4. each $\mathcal{C}_g$ has finitely many simple objects (up to iso)
5. the monoidal unit $\mathbb{1}$ simple

Ex:
- $\mathrm{Mod}_{fd}(A)$ with $A = \{A_g\}_{g \in G}$ semisimple of finite-type
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State sum invariants

A *n-type* is a connected CW-complex with $\pi_k = 0$ for $k > n$.

- **X is a 0-type:** $X \simeq \{\mathrm{pt}\}$ ($\Leftrightarrow$ G trivial)

Turaev-Viro (1992), Barret-Westburry (1996)

$\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}(M)$

- **X is a 1-type:** $X \simeq BG$ with G a group

Turaev-V. (2012)

$\mathcal{C}$ spherical G-fusion G-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}(M, h)$

Ex: $G\text{-}\mathrm{vect}_{\Bbbk}^{\omega}$ $\leftsquigarrow$ $\theta = [\omega] \in H^3(G, \Bbbk^*) \cong H^3(BG, \Bbbk^*)$ $\rightsquigarrow$ τ^{θ}

$$\mathrm{HTV}_{G\text{-}\mathrm{vect}_{\Bbbk}^{\omega}}(M, h) = \tau^{\theta}(M, h)$$

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Turaev-V. (2012)

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- **X is a 2-type:** $X \simeq B\mathcal{G}$ with $\mathcal{G}$ a 2-group

Sözer-V. (2022)

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2-groups

A 2-group is a monoidal category $\mathcal{G}$ such that:

- every morphism is invertible
- for $x \in \mathcal{G}$, there is $x^* \in \mathcal{G}$ with $x \otimes x^* \cong \mathbb{1} \cong x^* \otimes x$

Modelization:

A crossed module is $\begin{cases} \text{a group morphism } \chi\colon E \to H \\ \text{a left action of } H \text{ on } E \end{cases}$

such that $\chi({}^h e) = h\chi(e)h^{-1}$ and ${}^{\chi(e)}f = efe^{-1}$

Strictification: $\mathcal{G} \simeq_{\otimes} \mathcal{G}^{\text{strict}} \rightsquigarrow$ crossed module $\chi\colon E \to H$

$$H = \mathrm{Ob}(\mathcal{G}^{\text{strict}}) \quad E = \{\text{morphisms of source } \mathbb{1}\} \quad \chi = \text{target map}$$

Reconstruction: crossed module $\chi\colon E \to H \rightsquigarrow$ strict 2-group $\mathcal{G}_\chi$

$$\mathrm{Ob}(\mathcal{G}_\chi) = H \qquad \mathrm{Hom}_{\mathcal{G}_\chi}(x, y) = \left\{ e \in E \,|\, y = \chi(e)x \right\}$$

$$(y \xrightarrow{f} z) \circ (x \xrightarrow{e} y) = (x \xrightarrow{fe} z) \quad (x \xrightarrow{e} y) \otimes (z \xrightarrow{f} t) = (xz \xrightarrow{e^x f} yt)$$

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$$(y \xrightarrow{f} z) \circ (x \xrightarrow{e} y) = (x \xrightarrow{fe} z) \quad (x \xrightarrow{e} y) \otimes (z \xrightarrow{f} t) = (xz \xrightarrow{e^{x}f} yt)$$

2-groups

A 2-group is a monoidal category $\mathcal{G}$ such that:

- every morphism is invertible
- for $x \in \mathcal{G}$, there is $x^* \in \mathcal{G}$ with $x \otimes x^* \cong \mathbb{1} \cong x^* \otimes x$

<u>*Modelization:*</u>

A crossed module is $\begin{cases} \text{a group morphism } \chi\colon E \to H \\ \text{a left action of } H \text{ on } E \end{cases}$

such that $\chi({}^h e) = h\chi(e)h^{-1}$ and ${}^{\chi(e)}f = efe^{-1}$

Strictification: $\mathcal{G} \simeq_{\otimes} \mathcal{G}^{\text{strict}} \rightsquigarrow$ crossed module $\chi\colon E \to H$

$H = \mathrm{Ob}(\mathcal{G}^{\text{strict}})$ $E = \{\text{morphisms of source } \mathbb{1}\}$ $\chi = \text{target map}$

Reconstruction: crossed module $\chi\colon E \to H \rightsquigarrow$ strict 2-group $\mathcal{G}_\chi$

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Crossed modules and 2-types

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Examples:

- The inclusion $E \hookrightarrow H$ of a normal subgroup
- Any epimorphism $E \twoheadrightarrow H$ with central kernel
- $E \to \mathrm{Aut}(E)$ sending $e \in E$ to the inner automorphism
- $\partial\colon \pi_2(X, A, *) \to \pi_1(A, *)$ with $* \in A \subset X$

The classifying space $B\chi$ is a 2-type:

$$\pi_1(B\chi) = \mathrm{Coker}(\chi), \quad \pi_2(B\chi) = \mathrm{Ker}(\chi), \quad \pi_k(B\chi) = 0 \quad \text{for } k \geq 3$$

MacLane-Whitehead (1950)

Crossed modules model all connected homotopy 2-types

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Crossed modules model all connected homotopy 2-types

Monoidal categories graded by $\chi\colon E \to H$

A $\Bbbk$-linear monoidal category $\mathcal{C}$ is χ-graded if:

1. Hom-spaces are E-graded:

   $$\mathrm{Hom}_{\mathcal{C}}(X, Y) = \bigoplus_{e \in E} \mathrm{Hom}^{e}_{\mathcal{C}}(X, Y)$$

   For homogenous morphisms: $|\beta \circ \alpha| = |\beta|\,|\alpha|$ $\quad |\mathrm{id}_X| = 1$

2. Associator and unitors are of degree 1

   $\rightsquigarrow$ the monoidal subcategory $\mathcal{C}^1$ of degree 1 morphisms:

   $$\mathrm{Ob}(\mathcal{C}^1) = \mathrm{Ob}(\mathcal{C}) \qquad \mathrm{Hom}_{\mathcal{C}^1}(X, Y) = \mathrm{Hom}^{1}_{\mathcal{C}}(X, Y)$$

3. The subcategory $\mathcal{C}^1$ is H-graded

   $\rightsquigarrow$ $\mathcal{C}$ has homogeneous objects with degree in H

4. For homogenous objects: $\mathrm{Hom}^{e}_{\mathcal{C}}(X, Y) \neq 0 \Rightarrow |Y| = \chi(e)\,|X|$

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Monoidal categories graded by $\chi \colon E \to H$

A $\Bbbk$-linear monoidal category C is χ-graded if:

1. Hom-spaces are E-graded:

   $$\mathrm{Hom}_C(X,Y) = \bigoplus_{e \in E} \mathrm{Hom}_C^e(X,Y) \leftarrow \text{degree } e \text{ homogeneous morphisms}$$

   For homogenous morphisms: $|\beta \circ \alpha| = |\beta|\,|\alpha|$ $\quad |\mathrm{id}_X| = 1$

2. Associator and unitors are of degree 1

   $\rightsquigarrow$ the monoidal subcategory C^1 of degree 1 morphisms:

   $$\mathrm{Ob}(C^1) = \mathrm{Ob}(C) \qquad \mathrm{Hom}_{C^1}(X,Y) = \mathrm{Hom}_C^1(X,Y)$$

3. The subcategory C^1 is H-graded

   $\rightsquigarrow$ C has homogeneous objects with degree in H

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Monoidal categories graded by $\chi \colon E \to H$

A $\Bbbk$-linear monoidal category $\mathcal{C}$ is χ-graded if:

1. Hom-spaces are E-graded:

   $$\mathrm{Hom}_{\mathcal{C}}(X, Y) = \bigoplus_{e \in E} \mathrm{Hom}_{\mathcal{C}}^{e}(X, Y) \leftarrow \text{degree } e \text{ homogeneous morphisms}$$

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A χ-fusion category is a χ-graded category $\mathcal{C}$ such that:

1. the H-graded subcategory $\mathcal{C}^1$ is H-fusion
2. for all $e \in E$, each object X of $\mathcal{C}$ is a e-direct sum of simple objects of $\mathcal{C}^1$ $\quad (X = \bigoplus_\alpha s_\alpha \quad \text{with} \quad |s_\alpha \hookrightarrow X| = e)$

$\rightsquigarrow$ for a simple objet s of $\mathcal{C}^1$: $\left(\mathrm{Hom}^e_{\mathcal{C}}(s, X)\right)^* \cong \mathrm{Hom}^{e^{-1}}_{\mathcal{C}}(X, s)$

Ex: the linearization $\Bbbk\mathcal{G}_\chi$ of the 2-group $\mathcal{G}_\chi$ is χ-fusion

$$\mathrm{Ob}(\Bbbk\mathcal{G}_\chi) = H \quad \text{and} \quad \mathrm{Hom}^e_{\Bbbk\mathcal{G}_\chi}(x, y) = \begin{cases} \Bbbk & \text{if } y = \chi(e)x \\ 0 & \text{otherwise} \end{cases}$$

Twisting $\mathcal{C}$ by a 3-cocycle $\omega\colon H^3 \times E^3 \to \Bbbk^*$ for $\chi \quad \rightsquigarrow \quad \chi$-fusion $\mathcal{C}^\omega$

$$\omega\big(x, y, z, a, b, c\big)\,\omega\big(x, \chi(c)yz, t, ba^x(c^{-1}), f, d\big)\,\omega\big(y, z, t, c, d, e\big)$$
$$= \omega\big(\chi(a)xy, z, t, b, f, e\big)\,\omega\big(x, y, \chi(e)zt, a, fba^{xy}(e^{-1})a^{-1}, dc^y(e^{-1})\big)$$

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$$\omega\big(x,y,z,a,b,c\big)\,\omega\big(x,\chi(c)yz,t,ba^x(c^{-1}),f,d\big)\,\omega\big(y,z,t,c,d,e\big)$$
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A χ-fusion category is a χ-graded category $\mathcal{C}$ such that:

1. the H-graded subcategory $\mathcal{C}^1$ is H-fusion ($\mathcal{C}$ not semisimple)
2. for all $e \in E$, each object X of $\mathcal{C}$ is a e-direct sum of simple objects of $\mathcal{C}^1$ $\left(X = \bigoplus_\alpha s_\alpha \text{ with } |s_\alpha \hookrightarrow X| = e\right)$

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χ-fusion categories

A χ-fusion category is a χ-graded category $\mathcal{C}$ such that:

1. the H-graded subcategory $\mathcal{C}^1$ is H-fusion ($\mathcal{C}$ not semisimple)
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Ex: the linearization $\Bbbk\mathcal{G}_\chi$ of the 2-group $\mathcal{G}_\chi$ is χ-fusion

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$\mathcal{C}$ spherical χ-fusion category ⇝ trace of degree 1 endomorphisms

The graded 6j-symbol associated with a 6-tuple (i, j, k, ℓ, m, n) of objects and $(e_1, e_2, e_3, e_4) \in E^4$ such that $e_1 e_2 ({}^{|\ell|}e_3) e_4 = 1$ is

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More generally: isotopy invariant $F_{\mathcal{C}}$ of $\mathcal{C}$-colored χ-graphs in S^2

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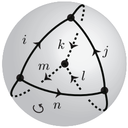

State sum invariants of 2-bundles over 3-manifolds

$\chi : E \to H$ crossed module, $\mathcal{C}$ spherical χ-fusion category

M closed oriented 3-manifold, $h \in [M, B\chi]$

Pick a triangulation of M and $g \in h$ with

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c= χ**-coloring** of the edges : c_e simple object of degree $[g(e)]$

$$\mathrm{HTV}_{\mathcal{C}}(M, h) = \sum_{c} \left(\prod_{e} \dim(c_e) \right) \mathrm{ctr}_f\left(\otimes_\Delta \ |\Delta| \right) \in \Bbbk$$

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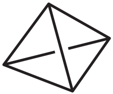

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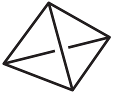

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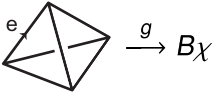

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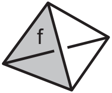

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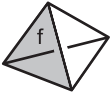

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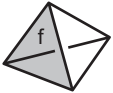

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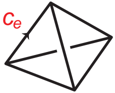

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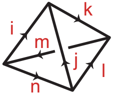

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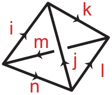

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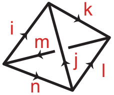

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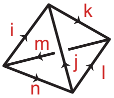

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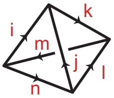

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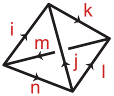

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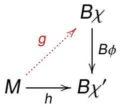

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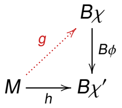

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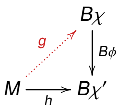

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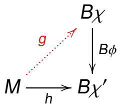

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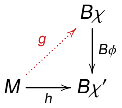

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Particular cases:

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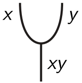
$\Delta_{x,y} =$ 

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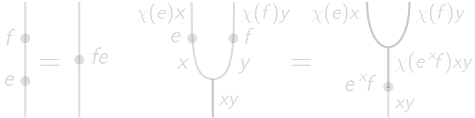

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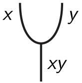
$\Delta_{x,y} =$ 

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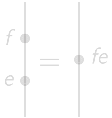

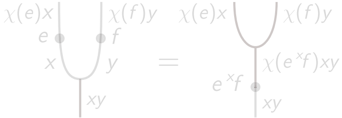

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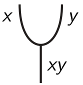
$\Delta_{x,y} =$ 

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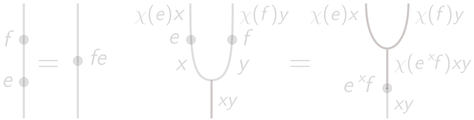

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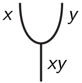
$\Delta_{x,y} =$ 

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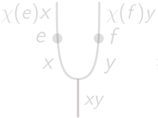

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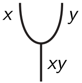
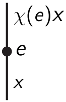
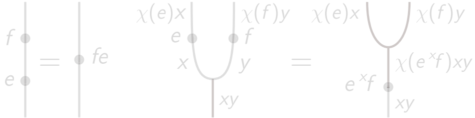
$\Delta_{x,y} =$ 

$\phi_{x,e} =$ 



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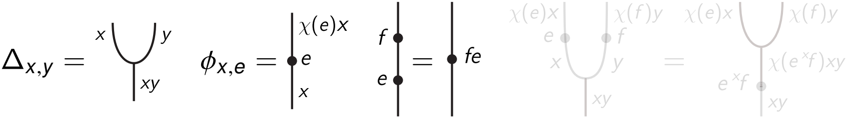

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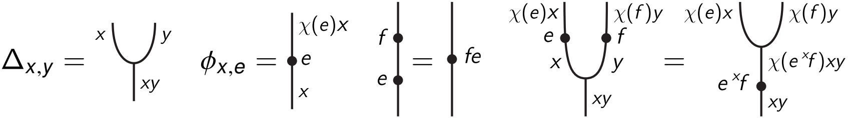

Particular cases:

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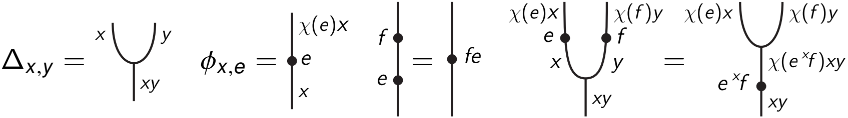

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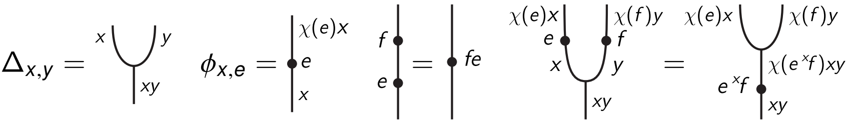

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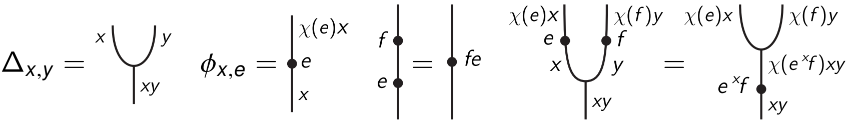

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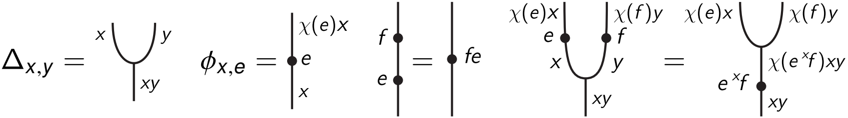

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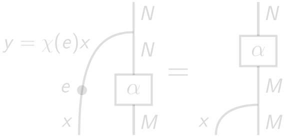

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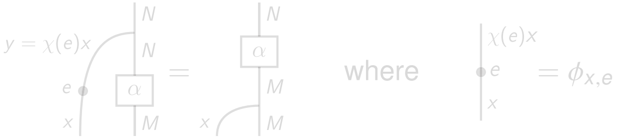

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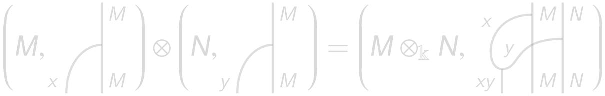

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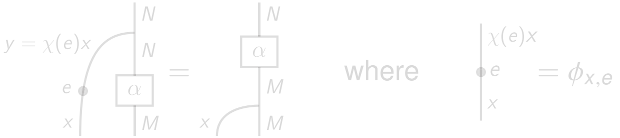

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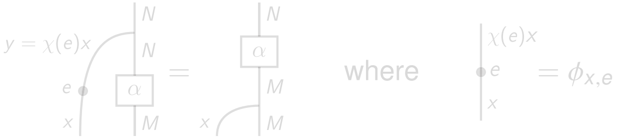

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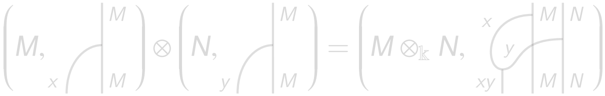

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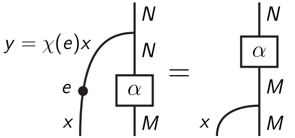

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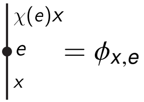

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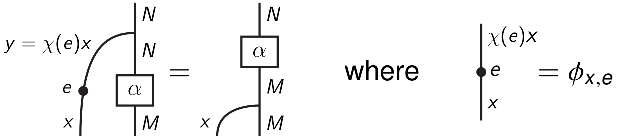

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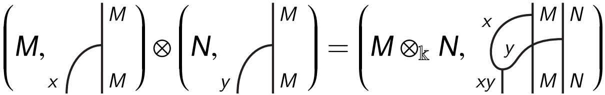

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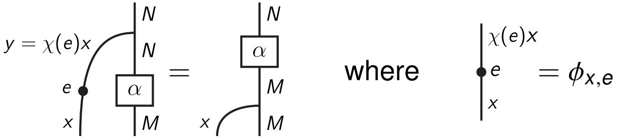

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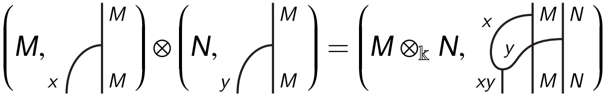

Example of a Hopf χ-coalgebra

$\mathbb{Z}/4\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\} \xrightarrow{\chi} \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ $\chi(\bar{n}) = 0$ ${}^{1}\bar{n} = -\bar{n}$

Consider: $A_0 = \mathbb{C}\langle a \,|\, a^4 = 1 \rangle$
$A_1 = \mathbb{C}\langle u, v \,|\, u^2 = 1 = v^2,\ vu = -uv \rangle$

with coproduct:

$$\Delta_{0,0}(a) = (a \otimes a)\Omega(a^2, a^2) \qquad \Delta_{1,1}(a) = (u \otimes u)\Omega(-v, v)$$
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$A_1 = \mathbb{C}\langle u, v \,|\, u^2 = 1 = v^2,\ vu = -uv\rangle \cong M_2(\mathbb{C})$

with coproduct:

$$\Delta_{0,0}(a) = (a \otimes a)\Omega(a^2, a^2) \qquad \Delta_{1,1}(a) = (u \otimes u)\Omega(-v, v)$$

$$\Delta_{0,1}(u) = (a \otimes u)\Omega(a^2, v) \qquad \Delta_{0,1}(v) = a^2 \otimes v$$

$$\Delta_{1,0}(u) = (u \otimes a)\Omega(-v, a^2) \qquad \Delta_{1,0}(v) = v \otimes a^2$$

$$\Omega(s, t) = \frac{1}{2}(1 \otimes 1 + s \otimes 1 + 1 \otimes t - s \otimes t)$$

and χ-action: $\phi_{0,\bar{n}}(a) = (-1)^n a \quad \phi_{1,\bar{n}}(u) = (-1)^n u \quad \phi_{1,\bar{n}}(v) = v$

$A = \{A_0, A_1\}$ is a Hopf χ-coalgebra which is involutory of finite type

$\Rightarrow \quad \mathrm{Mod}_{fd}(A)$ is a spherical χ-fusion category