Clifford deformations and Generalized Knörrer's Periodicity Theorem

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Outline

(I) Noncommutative quadric hypersurfaces

(II) Clifford deformations

(III) Generalized Knörrer's periodicity theorem

(IV) Isolated singularities from skew polynomial algebras

(I) Noncommutative quadric hypersurfaces

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- Let A be a connected graded algebra, i.e., $A_0 = \Bbbk$.

  A is called a Koszul algebra if the trivial module $\Bbbk_A$ has a graded free resolution

$$0 \longleftarrow \Bbbk_A \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \cdots \longleftarrow P_n \longleftarrow \cdots$$

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- **Example.** $A = \Bbbk[x, y]$,
  $A = \Bbbk_{-1}[x, y]$,
  $A = \Bbbk\langle x, y\rangle/(xy - yx + x^2)$.

Quantum polynomial algebra

- A noetherian connected graded algebra A is called an Artin-Schelter Gorenstein algebra if
  (1) $\text{injdim}_A A = \text{injdim} A_A = d < \infty$
  (2) $\text{Ext}^n_A(\Bbbk, A) = 0$ if $n \neq d$, and $\text{Ext}^d_A(\Bbbk, A) \cong \Bbbk$.

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- A quantum polynomial algebra is a Koszul Artin-Schelter regular algebra A such that
  (1) $H_A(t) = (1-t)^{-n}$ for some $n \geq 1$,
  (2) A is a domain.

  $H_A(t) = \sum_{n \geq 0} t^n \dim A_n$.

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  (1) A_f is a Koszul algebra.

  (2) If A is of global dimension d, then A_f is an Artin-Schelter Gorenstein algebra of injective dimension $d-1$.

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- $A = \mathbb{k}[x, y]$, $f = x^2 + y^2$, A_f
  $A' = \mathbb{k}_{-1}[x, y]$, $f = x^2 + y^2$, A'_f

MCM modules

- R a noetherian connected graded algebra.

  $\mathsf{gr}\, R$, category of finitely generated graded right R-modules

  $\mathsf{tors}\, R$, full subcategory of $\mathsf{gr}\, R$ consisting of finite dimensional modules

  $\mathsf{qgr}\, R = \mathsf{gr}\, R / \mathsf{tors}\, R$.

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  $\mathsf{qgr}\, R = \mathsf{gr}\, R / \mathsf{tors}\, R$.

- For $M \in \mathsf{gr}\, R$, let
  $\Gamma(M) = \{m \in M | mA$ is finite dimensional$\}$.

  The i-th right derived functor of Γ is denoted by $R^i\Gamma$.

  For $M \in \mathsf{gr}\, R$, the *depth* of M is defined to be the number

$$\mathsf{depth}(M) = \min\{i | R^i\Gamma(M) \neq 0\}.$$

MCM modules

- Suppose that R is an Artin-Schelter Gorenstein algebra with injdimR_R=injdim$_R R = d$.

  $M \in \mathsf{gr}\, R$ is called a maximal Cohen-Macaulay module (MCM module) if depth$(M) = d$.

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- $\mathsf{mcm}\, R$ the category of all the MCM over R.

  $\mathsf{mcm}\, R$ is a Frobenius category, hence the stable category $\underline{\mathsf{mcm}} R$ is a triangulated category.

  The category $\underline{\mathsf{mcm}} R$ is sometimes called the singularity category of R.

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- **Example.** $A = \Bbbk[x, y]$, $A' = \Bbbk_{-1}[x, y]$, $f = x^2 + y^2$.
  $\underline{\mathrm{mcm}} A_f \cong \underline{\mathrm{mcm}} A'_f \cong D^b(\Bbbk \times \Bbbk)$.

MCM modules

- A fundamental result:

Theorem

Let A be a quantum polynomial algebra and let $f \in A_2$ be a central element. Then there is a finite dimensional algebra $C(A_f)$ such that there is an equivalence of triangulated categories

$$D^b(C(A_f)) \cong \underline{\mathrm{mcm}} A_f.$$

S. P. Smith, M. Van den Bergh, Noncommutative quadric surfaces, J. Noncommut. Geom. 7 (2013), 817–856.

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- **Problems:**
  (1) find a way to compute $C(A_f)$;
  (2) Let $A = \Bbbk[x, y]$ and $A' = \Bbbk_{-1}[x, y]$, and let $f = x^2 + y^2$. Note that $C(A_f) \cong C(A'_f) \cong \Bbbk \times \Bbbk$. So, how can we recognize the difference between A_f and A'_f?

(II) Clifford deformations

Clifford deformation of Koszul algebra

- Let V be a finite dimensional vector space, and let $E = T(V)/(R)$ be a Koszul algebra, where $R \subseteq V \otimes V$.

  A linear map $\theta : R \to \Bbbk$ is called a Clifford map if

$$(\theta \otimes 1 - 1 \otimes \theta)(V \otimes R \cap R \otimes V) = 0.$$

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- Note that a Clifford deformation is a special case of Poicaré-Birkhoff-Witt deformations.
- The usual Clifford algebra

  $$\mathbb{R}_n^{p,q} = \mathbb{R}\langle x_1, \ldots, x_n \rangle / (x_i^2 + 1, x_j^2 - 1 : 1 \leq i \leq p, p+1 \leq j \leq p+q)$$

  is a Clifford deformation of the exterior algebra $E = \bigwedge\{x_1, \ldots, x_n\}$.

Clifford deformation of Koszul algebra

- Let A be a quantum polynomial algebra.

Proposition

Let $E = A^!$ be the quadratic dual of the quantum polynomial algebra A. Then E is a Koszul Frobenius algebra.

S.P. Smith, Some finite dimensional algebras related to elliptic curves, in: CMS Conf. Proc., 1996

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- We have the following facts:

Proposition

- *Each central element $0 \neq f \in A_2$ is corresponding to a Clifford map θ_f of $E = A^!$.*
- *The Clifford deformation $E(\theta_f)$ is a strongly $\mathbb{Z}_2$-graded algebra.*
- $C(A_f) \cong E(\theta_f)_0$.

Example

- Let $A = \Bbbk\langle x, y, z\rangle/(r_1, r_2, r_3)$, where $r_1 = zx + xz$, $r_2 = yz + zy$, $r_3 = x^2 + y^2$. Then A is a quantum polynomial algebra of dimension 3.

f	$C(A_f) = E(\theta_f)_0$
$z^2 + xy + yx + \lambda x^2$	$\mathbf{k}^4$
$z^2 + xy + yx \pm 2\sqrt{-1}x^2$	$\mathbf{k}[u]/(u^2) \times \mathbf{k}[u]/(u^2)$
z^2	$\mathbf{k}[u, v]/(u^2 - v^2, uv)$
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- **Remark.** Let A be a quantum polynomial algebra of dimensional 3. If $f \in A_2$ is a central element, $A_f = A/Af$ is called a noncommutative conic. The algebras $C(A_f)$ have been classified for noncommutative conics.

H. Hu, Classification of noncommutative conics associated to symmetric regular superpotentials, J. Algebra Appl. 22 (2023), 2350136.

H. Hu, M. Matsuno, I. Mori, Noncommutative conics in Calabi-Yau quantum planes, J. Algebra 620 (2023), 194–224.

Theorem

Let A be a quantum polynomial algebra, and let $f \in A_2$ be a central regular element.

Then qgr A_f *has finite global dimension (i.e., projA_f is smooth) if and only if $C(A_f) = E(\theta_f)_0$ is a semisimple algebra.*

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J.-W. He, Y. Ye, Clifford deformations of Koszul Frobenius algebras and noncommutative quadrics, Algebra Colloq. 2024.

I. Mori, K. Ueyama, Noncommutative Knörrer Periodicity Theorem and noncommutative quadric hypersurfaces, Algebra Number Theory, 2022.

(III) Generalizations of Knörrer's periodicity theorem

An example

- **Example.** $A = \Bbbk[x,y]$, $A' = \Bbbk_{-1}[x,y]$, $f = x^2 + y^2$.

  Then $\underline{\mathsf{mcm}}(A_f) \cong \underline{\mathsf{mcm}}(A'_f) \cong D^b(\Bbbk \times \Bbbk)$.

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- Let B be a quantum polynomial algebra and $g \in B_2$ be a central element.

  Consider the tensor algebra $B \otimes A$ and $B \otimes A'$,

  let $h := g \otimes 1 + 1 \otimes f \in B \otimes A$ (or in $B \otimes A'$, resp.).

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- Fact: $\underline{\mathsf{mcm}}(B \otimes A)_h$ is different from $\underline{\mathsf{mcm}}(B \otimes A')_h$!

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- Fact: $\underline{\mathsf{mcm}}(B \otimes A)_h$ is different from $\underline{\mathsf{mcm}}(B \otimes A')_h$!

- The reason is the following:

  Let E and E' be the Koszul dual of A and A' respectively.

  The Clifford deformation of E and that of E' associated to f are very different!

  Indeed, $E(\theta_f) \cong \mathbb{M}_2(\Bbbk)$ and $E'(\theta_f) \cong \Bbbk\mathbb{Z}_2 \times \Bbbk\mathbb{Z}_2$.

- It seems reasonable to classify the noncommutative quadric hypersurfaces according the Clifford deformations.

Definition

Let A be a quantum polynomial algebra, and let $f \in A_2$ be a central element. Let E be the Koszul dual of A.

If the Clifford deformation $E(\theta_f)$ is a simple $\mathbb{Z}_2$-graded algebra, then we call $A_f = A/Af$ is a simple graded isolated singularity.

$\mathbb{Z}_2$-gradings on matrix algebras

- Let $n = p + q$, and let $M_n(\Bbbk)$ be the matrix algebra over $\Bbbk$.

  One may define a $\mathbb{Z}_2$-grading on $M_n(\Bbbk)$:

$$M(p|q)_0 = \begin{bmatrix} M_p(\Bbbk) & 0 \\ 0 & M_q(\Bbbk) \end{bmatrix},$$

$$M(p|q)_1 = \begin{bmatrix} 0 & M_{p\times q}(\Bbbk) \\ M_{q\times p}(\Bbbk) & 0 \end{bmatrix}.$$

  Denoted by $M(p|q)$ the matrix algebra with the above $\mathbb{Z}_2$-grading.

Simple $\mathbb{Z}_2$-graded algebras

- There are two classes of simple $\mathbb{Z}_2$-graded algebra:

  (0) matrix algebras $M(p|q)$;

  (1) matrix algebras over $\Bbbk\mathbb{Z}_2$.

Simple $\mathbb{Z}_2$-graded algebras

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  (0) matrix algebras $M(p|q)$;

  (1) matrix algebras over $\Bbbk\mathbb{Z}_2$.

- If the $\mathbb{Z}_2$-graded algebra $E(\theta_f)$ is isomorphic to $M(p|q)$, then we call A_f a simple graded isolated singularity of type-0

  If the $\mathbb{Z}_2$-graded algebra $E(\theta_f)$ is isomorphic to a matrix algebra over $\Bbbk\mathbb{Z}_2$, then we call A_f a simple graded isolated singularity of type-1.

- **Proposition.** Let A and B be quantum polynomial algebras, and let $f \in A_2$ and $g \in B_2$ be central elements. Suppose that $A \otimes B$ is noetherian. Let $h = f \otimes 1 + 1 \otimes g \in A \otimes B$.
    - If both A_f and B_f are simple graded isolated singularity of type-1, then $(A \otimes B)_h$ is a simple graded isolated singularity of type-0.
    - If A_f is a simple graded isolated singularity of type-1 and B is a simple graded isolated singularity of type-0, then$(A \otimes B)_h$ is a simple graded isolated singularity of type-1.
    - If both A_f and B_f are simple graded isolated singularity of type-0, then $(A \otimes B)_h$ is a simple graded isolated singularity of type-0.

- A key lemma.

Lemma

Let A and B be quantum polynomial algebras, and let $f \in A_2$ and $g \in B_2$ be central elements. Suppose that $A \otimes B$ is noetherian. Let $h = f \otimes 1 + 1 \otimes g \in A \otimes B$.

Then we have an isomorphism of $\mathbb{Z}_2$-graded algebras

$$E_{(A \otimes B)^!}(\theta_h) \cong E_{A^!}(\theta_f) \hat{\otimes} E_{B^!}(\theta_g),$$

where $\hat{\otimes}$ is the $\mathbb{Z}_2$-graded tensor.

- Let $A = \Bbbk[x, y]$, and $f = x^2 + y^2$. Then

$$E(\theta_f) \cong \mathbb{M}_2(\Bbbk),$$

where $\mathbb{M}_2(\Bbbk)$ is viewed as a $\mathbb{Z}_2$-graded algebra by setting

$$\mathbb{M}_2(\Bbbk)_0 = \begin{bmatrix} \Bbbk & 0 \\ 0 & \Bbbk \end{bmatrix}, \quad \mathbb{M}_2(\Bbbk)_1 = \begin{bmatrix} 0 & \Bbbk \\ \Bbbk & 0 \end{bmatrix}.$$

Hence A_f is a simple graded isolated singularity of type-0.

Examples

- Let $A = \Bbbk[x, y]$, and $f = x^2 + y^2$. Then
$$E(\theta_f) \cong \mathbb{M}_2(\Bbbk),$$
where $\mathbb{M}_2(\Bbbk)$ is viewed as a $\mathbb{Z}_2$-graded algebra by setting
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Hence A_f is a simple graded isolated singularity of type-0.
- Let $A = \Bbbk\langle x_1, \ldots, x_5\rangle/(r_1, \ldots, r_{10})$, where the generating relations are as follows:
$$r_1 = x_1x_2 - x_2x_1, r_2 = x_1x_3 + x_3x_1, r_3 = x_1x_4 + x_4x_1,$$
$$r_4 = x_1x_5 + x_5x_1, r_5 = x_2x_3 - x_3x_2, r_6 = x_2x_4 + x_4x_1,$$
$$r_7 = x_2x_5 + x_5x_2, r_8 = x_3x_4 - x_4x_3, r_9 = x_3x_5 + x_5x_3,$$
$$r_{10} = x_4x_5 + x_5x_4.$$
Let $f = x_1^2 + \cdots + x_5^2$. Then A_f is a simple graded isolated singularity of type-1.

- **Remark.** We are unable to find a way to characterize when A_f is a simple graded isolated singularity.

  A criterion for singularities obtained from skew polynomial algebras to be simple graded isolated singularity will be given at the next part of the talk.

- We have the following generalized Knörrer's periodicity theorem.

Theorem

Let A and B be quantum polynomial algebras, and let $f \in A_2$ and $g \in B_2$ be central elements. Suppose that $A \otimes B$ is noetherian, and let $h = f \otimes 1 + 1 \otimes g \in A \otimes B$.

(i) *If B_g is a simple graded isolated singularity of tpye-0, then there are equivalences of triangulated categories*
$$\underline{\mathrm{mcm}}(A \otimes B)_h \cong D^b(\mathrm{mod} E(\theta_f)_0) \cong \underline{\mathrm{mcm}} A_f;$$

(ii) *If B_g is a simple graded isolated singularity of type-1, there is an equivalence of triangulated categories*
$$\underline{\mathrm{mcm}}(A \otimes B)_h \cong D^b(\mathrm{mod} E(\theta_f)^\natural),$$
where $E(\theta_f)^\natural$ is the underlying ungraded algebra, and $\mathrm{mod} E(\theta_f)^\natural$ is the category of all the finite dimensional modules over $E(\theta_f)^\natural$.

J.-W. He, X.-C. Ma, Y. Ye, Clifford deformations and Generalized Knörrer Periodicity Theorem, preprint, 2025.

Noncommutative Knörrer's periodicity theorem

- In particular, if we take $B = \Bbbk[x, y]$ and $g = x^2 + y^2$, then we obtain:

Theorem

Let A be a quantum polynomial algebra and let $f \in A_2$ be a central element. Let $A_f^{\#\#} = A[x, y]/(f + x^2 + y^2)$. Then

$$\underline{\mathrm{mcm}} A_f^{\#\#} \cong \underline{\mathrm{mcm}} A_f.$$

H. Knörrer, Cohen-Macaulay modules on hypersurface singularities I, Invent. Math. 88 (1987), 153–164.

A. Conner, E. Kirkman, W. F. Moore, C. Walton, Noncommutative Knörrer periodicity and noncommutative Kleinian singularities, J. Algebra 540 (2019), 234–273.

J.-W. He, Y. Ye, Clifford deformations of Koszul Frobenius algebras and noncommutative quadrics, Algebra Colloq. 2024.

I. Mori, K. Ueyama, Noncommutative Knörrer's Periodicity Theorem and noncommutative quadric surfaces, J. Algebra 611(2022), 528–560.

(IV) Isolated singularities obtained from skew polynomial algebras

- Let $A = \Bbbk_{\varepsilon_{ij}}[x_1, \ldots, x_n] = \Bbbk\langle x_1, \ldots, x_n\rangle / (x_i x_j - \varepsilon_{ij} x_j x_i)$, where $\varepsilon_{ij} = \pm 1$.

  $f = x_1^2 + \cdots + x_n^2 \in A_2$.

Graphs of skew polynomial algebras

- Let $A = \Bbbk_{\varepsilon_{ij}}[x_1, \dots, x_n] = \Bbbk\langle x_1, \dots, x_n\rangle/(x_i x_j - \varepsilon_{ij} x_j x_i)$, where $\varepsilon_{ij} = \pm 1$.

  $f = x_1^2 + \cdots + x_n^2 \in A_2$.

- Define a graph G_A associated to A as following:

$$V(G_A) = \{1, \dots, n\}$$

$$i \sim j \text{ if and only if } \varepsilon_{ij} = 1.$$

Graphs of skew polynomial algebras

- Let $A = \Bbbk_{\varepsilon_{ij}}[x_1, \dots, x_n] = \Bbbk\langle x_1, \dots, x_n\rangle/(x_ix_j - \varepsilon_{ij}x_jx_i)$, where $\varepsilon_{ij} = \pm 1$.

  $f = x_1^2 + \cdots + x_n^2 \in A_2$.

- Define a graph G_A associated to A as following:

$$V(G_A) = \{1, \dots, n\}$$

$$i \sim j \text{ if and only if } \varepsilon_{ij} = 1.$$

- graph Clifford algebra C_{G_A}, which is generated by elements $e_1, \dots, e_n$ with relations

$$\begin{aligned} e_ie_j &= e_je_i, &\quad \text{if} &\quad i \nsim j;\\ e_ie_j &= -e_je_i, &\quad \text{if} &\quad i \sim j;\\ e_i^2 &= 1, &\quad \text{for all} &\quad i. \end{aligned}$$

- **Facts.** (1) $A = \Bbbk_{\varepsilon_{ij}}[x_1, \ldots, x_n]$ is a Koszul algebra, and $f = x_1^2 + \cdots + x_n^2 \in A_2$ is a central element of A.

  (2) The Koszul dual of A is the Grassmann algebra E associated to the graph G_A.

  (2) The Clifford deformation of E defined by the central element f:

$$E(\theta_f) \cong C_{G_A}.$$

Edge reductions of graphs

Definition

Let G be a graph with nonempty edge sets. Assume $i \sim j$. Then the edge reduction of G (with respect to $i \sim j$) is a new graph $G' = \mu_{i,j}(G)$ with the same vertex set as G and the edge set obtained from $E(G)$ following the rules

(i) if $s \notin \{i,j\}$, then $s \nsim i$ and $s \nsim j$ in G';

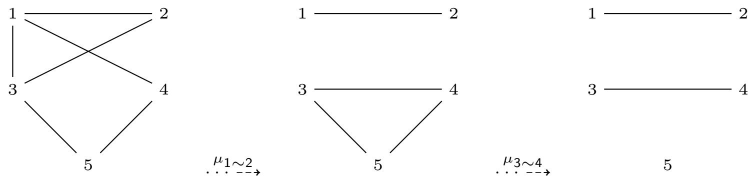
(ii) for any $s, t \notin \{i,j\}$, if one of the following conditions is satisfied:

1. $s \nsim i, s \nsim j$;
2. $t \nsim i, t \nsim j$;
3. $s \sim i, t \sim i, s \sim j, t \sim j$;
4. $s \sim i, t \sim i, s \nsim j, t \nsim j$;
5. $s \nsim i, t \nsim i, s \sim j, t \sim j$;

then $s \sim t$ in G' if and only if $s \sim t$ in G;

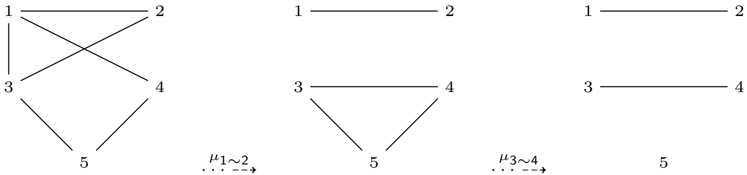
(iii) otherwise, $s \sim t$ in G' if and only if $s \nsim t$ in G.

- **Example.**



Edge reductions of graphs

- **Example.**



- **Fact.** Let G be a graph, and assume $i \sim j$. Let $G' = \mu_{i\sim j}(G)$. Then $C_G \cong C_{G'}$.

  In general, the graph Clifford algebras C_G and $C_{G'}$ are not isomorphic as $\mathbb{Z}_2$-graded algebras.

Isolated singularities of type-0

Proposition

Let $A = \Bbbk_{\varepsilon_{ij}}[x_1, \ldots, x_n]$ be the skew polynomial algebra, and $f = x_1^2 + \cdots + x_n^2 \in A$.

Then A_f is a simple graded isolated singularity of type-0 $\Longleftrightarrow$

there are a sequence of edge reductions so that the result graph is a union of

$$1 \text{ ——— } 2$$

J.-W. He, X.-C. Ma, Y. Ye, Clifford deformations and Generalized Knörrer Periodicity Theorem, preprint, 2025.

Isolated singularities of type-0

- For the graph G with n vertices, there is an adjacency matrix defined as follows: $M = (m_{ij})_{n\times n}$ with the entries

$$m_{ij} = \begin{cases} 1, & \text{if } i \sim j; \\ 0, & \text{if } i \nsim j. \end{cases}$$

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-

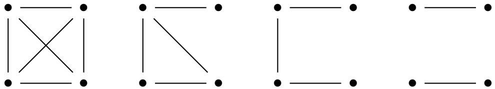
Proposition

Let $A = \Bbbk_{\varepsilon_{ij}}[x_1, \ldots, x_n]$ be the skew polynomial algebra. Let G_A be the associated graph of A, and let M be the adjacency matrix of G_A.

Then A_f is a simple graded isolated singularity of type-0 $\Longleftrightarrow$ $\det(M) = 1$ in $\mathbb{F}_2$.

J.-W. He, X.-C. Ma, Y. Ye, Clifford deformations and Generalized Knörrer Periodicity Theorem, preprint, 2025.

- **Example.**
  Let $A = \Bbbk_{\varepsilon_{ij}}[x_1, x_2, x_3, x_4]$ and $f = x_1^2 + x_2^2 + x_3^2 + x_4^2$. Then $A/(f)$ is a simple graded isolated singularity of type-0 if and only if the associated graph G_A is one of the following graphs:



Isolated singularities of type-1

Proposition

Let G be a graph with adjacency matrix $M = (g_{ij})_{n\times n}$. Then

(i) *If $C_G \cong M_m(\Bbbk\mathbb{Z}_2)$ for some m, then rank$(M) = n-1$ as a matrix in $\mathbb{F}_2$. In this case $n = 2k+1$ for some k and $m = 2^k$;*

(ii) *Assume that rank$(M) = n-1 = 2k$. Let $(a_1, \cdots, a_n) \in \mathbb{F}_2^n$ be the unique nonzero vector such that $(a_1, \cdots, a_n)M = 0$.*

Then $Z(C_G)$ is spanned by $\{1, c\}$ as a $\Bbbk$-vector space, where $c = e_1^{a_1} \cdots e_n^{a_n}$.

In particular, $C_G \cong M_m(\Bbbk\mathbb{Z}_2)$ if and only if $a_1 + \cdots + a_n \neq 0$ in $\mathbb{F}_2$.

J.-W. He, X.-C. Ma, Y. Ye, Clifford deformations and Generalized Knörrer Periodicity Theorem, preprint, 2025.

Isolated singularities of type-1

Theorem

Let $A = \Bbbk_{\varepsilon_{ij}}[x_1, \dots, x_n]$ *be a skew polynomial algebra, and* $f = x_1^2 + \cdots + x_n^2$. *Let* G_A *be the associated graph of* A *and* M *the adjacency matrix. Then*

A_f *is a simple graded isolated singularity of type-1 if and only if* n *is odd,* $\mathrm{rank} M = n - 1$, *and* $a_1 + \cdots + a_n \neq 0$ *in* $\mathbb{F}_2$, *where* $(a_1, \cdots, a_n)$ *is the unique nonzero vector such that* $(a_1, \cdots, a_n)M = 0$.

J.-W. He, X.-C. Ma, Y. Ye, Clifford deformations and Generalized Knörrer Periodicity Theorem, preprint, 2025.

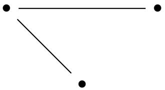
Examples

- Let $A = \Bbbk_{\varepsilon_{ij}}[x_1, x_2, x_3]$ and $f = x_1^2 + x_2^2 + x_3^2$. Then A_f is a simple graded isolated singularity of type 1 if and only if the associated graph G_A is one of the following graphs.

Examples

- Let $A = \Bbbk_{\varepsilon_{ij}}[x_1, x_2, x_3]$ and $f = x_1^2 + x_2^2 + x_3^2$. Then A_f is a simple graded isolated singularity of type 1 if and only if the associated graph G_A is one of the following graphs.

- Let $A = \Bbbk\langle x_1, x_2, x_3\rangle/(x_1x_2 + x_2x_1, x_1x_3 + x_3x_1, x_2x_3 - x_3x_2)$ and $f = x_1^2 + x_2^2 + x_3^2$. Then the associated graph G_A is given as follows.



As an $\mathbb{Z}_2$-graded algebra, $C_{G_A} \cong M(1|1) \times M(1|1)$ is not graded simple, and hence A_f is not a simple graded isolated singularity.

Thank you for your attention!