Towards Triangulated Birepresentations: Evaluation and Induction in Soergel Categories

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What is this talk about?

- Representation theory of algebras studies homomorphisms

$$A \longrightarrow \mathrm{End}(V).$$

- Representation theory of bicategories: actions on categories, i.e.

$$\mathcal{C} \longrightarrow \mathrm{End}(\mathcal{M}).$$

- These are called **2-representations.**

This talk is about 2-representations coming from affine Hecke algebras.

The story in one picture

affine Hecke algebras	$\longrightarrow$	Soergel categories
representations	$\longrightarrow$	2-representations
evaluation, induction	$\longrightarrow$	categorified evaluation, induction

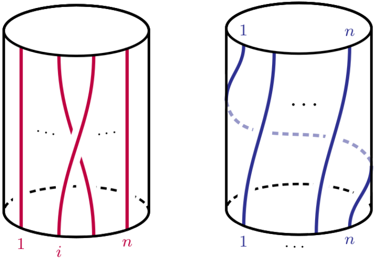
We start from affine Hecke algebras and their representations.

Why this problem?

- Finite-dimensional reps of affine Hecke algebras are rich and subtle.
- Two fundamental tools to study them:
  - evaluation representations,
  - parabolic (Zelevinsky) induction.
- Affine-type categorification leads to non-finitary settings:
  - infinitely many indecomposable 1-morphisms,
  - no finitary Hom-categories,
  - passing to homotopy / triangulated categories becomes unavoidable when looking at 2-representations. This is a new phenomenon.

Extended affine Hecke algebra $\widehat{H}_n^{\mathrm{ext}}$:

- Generators T_i and rotation $\rho^{\pm 1}$.
- Relations $\rho T_i \rho^{-1} = T_{i+1}$, $(T_i + q)(T_i - q^{-1}) = 0$, ...
- $H_n \subset \widehat{H}_n^{\mathrm{ext}}$ f.d. subalgebra generated by $T_1, \ldots, T_{n-1}$.

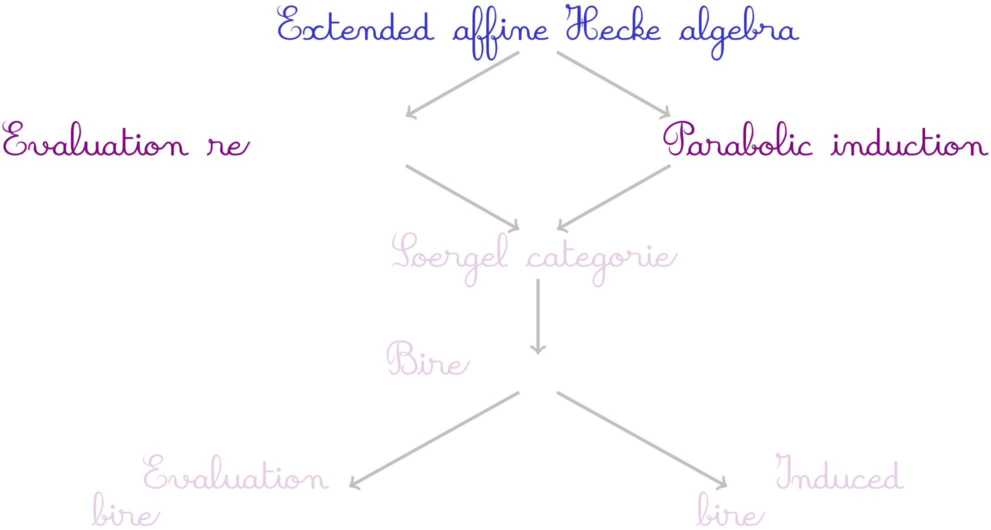


T_i

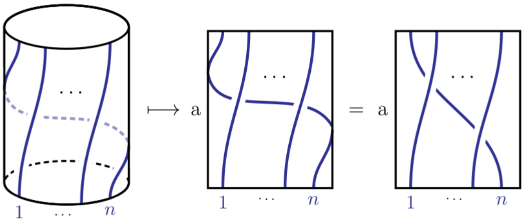
$(1 \le i \le n)$
indices mod n

ρ

Two constructions of representations

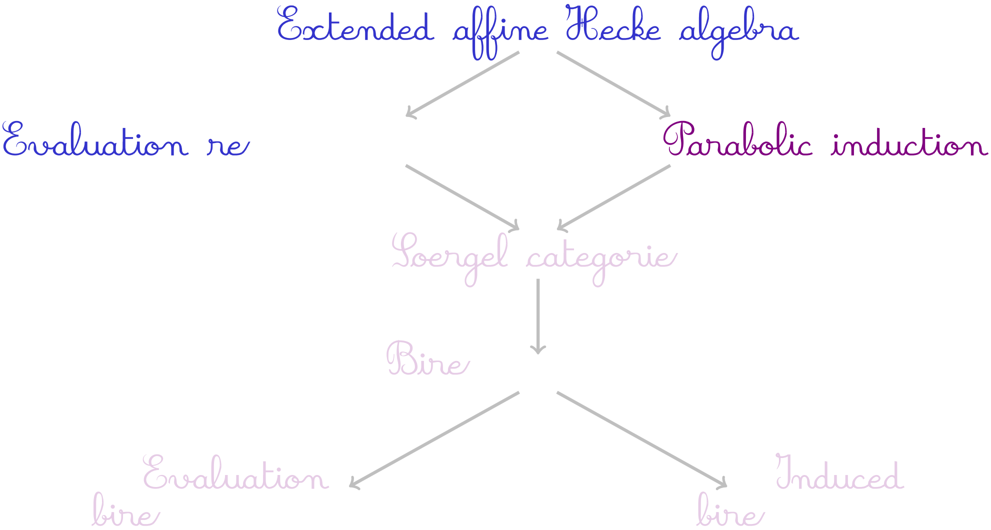


- For $a \in \mathbb{C}(q)^\times$: $\mathrm{ev}_a : \widehat{H}_n^{\mathrm{ext}} \longrightarrow H_n$
- Defined by $\begin{cases} T_i \mapsto T_i & (i \neq 0) \\ \rho \mapsto a T_{n-1} \cdots T_1 \end{cases}$



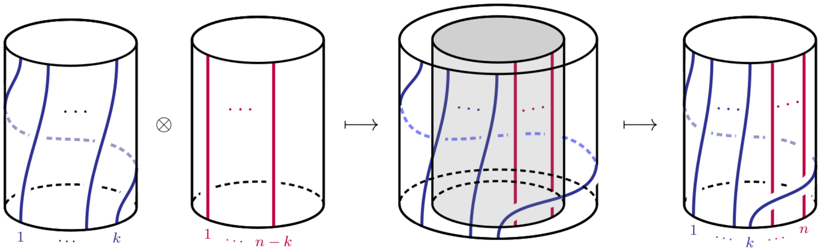
- Pullback produces evaluation representations.
- Key ingredient in finite-dimensional affine type A theory.

Two constructions of representations



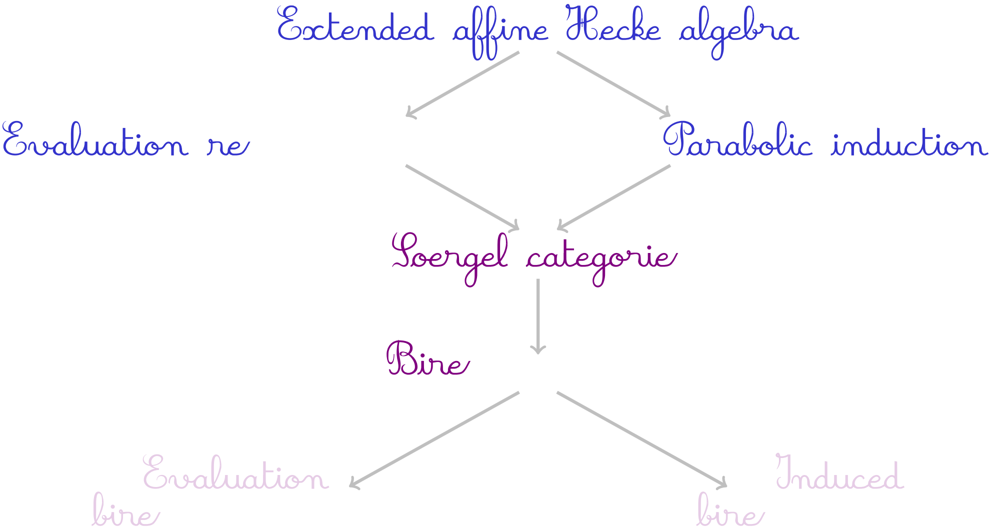
Classical parabolic induction (maximal parabolic)

- Decomposition $n = k + (n - k)$ gives a maximal parabolic $S_k \times S_{n-k} \subset S_n$.
- Induction via algebra embeddings: $\widehat{H}_k^{\text{ext}} \otimes \widehat{H}_{n-k}^{\text{ext}} \hookrightarrow \widehat{H}_n^{\text{ext}}$.



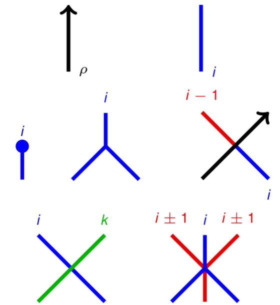
- Zelevinsky tensor product: $M_1 \odot M_2 := \mathrm{Ind}(M_1 \otimes M_2)$.

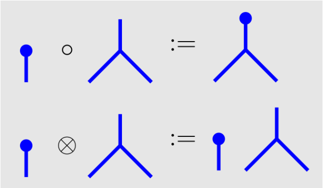
Categorification



Soergel categories $\mathcal{S}$

- Soergel categories categorify Hecke algebras :
  - They are $\mathbb{K}$-linear, additive, and monoidal (one-object bicategories).
  - $K_0(\mathcal{S}(W)) \cong H(W)$ (W Coxeter group).
  - Kahzdan–Lustig generators $b_i := T_i + q$.
  - B_i categorifies b_i
- Extended affine type A category $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$.
- Diagrammatic calculus encodes relations.





From representations to 2-representations

Soergel cats categorify Hecke algebras, so they must act on categories.

- Classical representation theory studies algebra homomorphisms

$$A \longrightarrow \mathrm{End}(V), \qquad (V \text{ a vector space})$$

- Categorify V to a category $\mathcal{M}$: replace $\mathrm{End}(V)$ by the bicategory $\mathrm{End}(\mathcal{M})$ of endofunctors and natural transformations.

- Formally, a *2-representation* of a bicategory $\mathcal{C}$ is a pseudofunctor

$$\mathcal{C} \longrightarrow \mathrm{End}(\mathcal{M}) \subset \mathsf{Cat}.$$

Terminology

- In this talk, a *birepresentation* of a bicategory $\mathcal{C}$ means a (wide) finitary 2-representation in the sense of Mazorchuk–Miemietz and Macpherson.
- For Soergel categories we will always work in this 2-representation framework, and later in its triangulated analogue.

From finitary to wide finitary

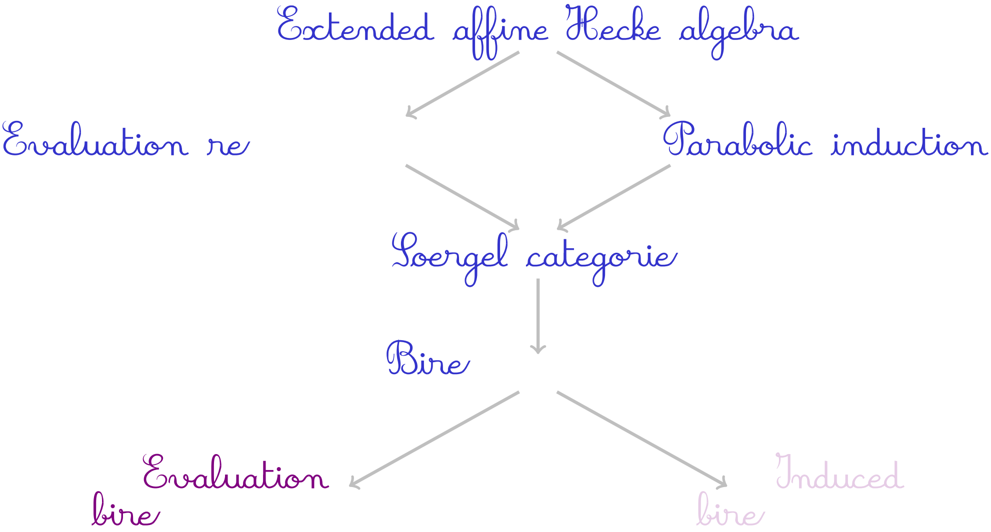
- *Finitary birepresentations* (Mazorchuk–Miemietz, 2011): 2-representations where all Hom-categories are equivalent to finitary additive categories.
  - Perfect for categorifying representations of finite-dimensional algebras.

In affine type, Soergel categories have infinitely many indecomposable 1-morphisms, so the finitary condition fails.

- We therefore use Macpherson's *wide finitary 2-representations* (2022), which allow infinitely many indecomposable 1-morphisms while keeping each Hom-category finitary.

- Evaluation and induction functors are built using Rouquier complexes.
- These live naturally in the homotopy category $K^b(\mathcal{S})$, not in the Soergel category $\mathcal{S}$ itself.

The correct framework four our 2-representations is triangulated rather than abelian or finitary


Extended affine Hecke algebra
Evaluation re
Parabolic induction
Soergel categorie
Bire
Evaluation bire
Induced bire

Goal: categorify ev_a.

- Monoidal functors: $\mathrm{Ev}_{r,s} \colon \widehat{\mathcal{S}}_n^{\mathrm{ext}} \to K^b(\mathcal{S}^{\mathrm{fin}})$.

  Uses Rouquier complexes $\mathcal{R}_i := \underline{\mathrm{B}_i} \xrightarrow{\bullet} R$ (R_i categorifies T_i)

$$\mathrm{Ev}_{r,s} \colon \begin{cases} \mathrm{B}_i \mapsto \mathrm{B}_i \quad (i \neq n) \\ \mathrm{B}_\rho \mapsto \mathrm{R}_{n-1} \otimes \cdots \otimes \mathrm{R}_1[r]\langle s \rangle \end{cases}$$

- Rouquier calculus only partially developed, but it suffices!

Evaluation functors: result

Theorem (Mackaay–Miemietz–Vaz, 2024)

For each (r, s) there is a monoidal functor

$$\mathrm{Ev}_{r,s} : \widehat{\mathcal{S}}_n^{\mathrm{ext}} \to K^b(\mathcal{S}_n)$$

categorifying the classical evaluation homomorphism.

What this gives:

- A family of triangulated 2-representations of $\widehat{\mathcal{S}}_n^{\mathrm{ext}}$.

Why it matters:

- These are the first evaluation 2-representations in affine type A.

We now do for parabolic induction what we did for evaluation...

Categorified parabolic induction (maximal parabolic)

- Categorical embeddings

$$\Psi_L \colon \widehat{\mathcal{S}}_k^{\text{ext}} \to K^b(\widehat{\mathcal{S}}_n^{\text{ext}}) \quad \text{and} \quad \Psi_R \colon \widehat{\mathcal{S}}_{n-k}^{\text{ext}} \to K^b(\widehat{\mathcal{S}}_n^{\text{ext}})$$

$$\Psi_L(\mathrm{B}_i) = \mathrm{B}_i \qquad \Psi_R(\mathrm{B}_j) = \mathrm{B}_{j+k}$$

$$\Psi_L(\mathrm{B}_\rho) = \mathrm{B}_\rho \mathrm{R}_{n-1}\mathrm{R}_{n-2}\cdots\mathrm{R}_k \qquad \Psi_R(\mathrm{B}_\rho) = \mathrm{R}_k^{-1}\cdots\mathrm{R}_2^{-1}\mathrm{R}_1^{-1}\mathrm{B}_\rho$$

- Symmetric braiding $\Psi_L\Psi_R \cong \Psi_R\Psi_L$

Categorified parabolic induction

Theorem (Mackaay–Miemietz–Vaz, 2025)

There is a monoidal functor

$$\Psi_{k,n-k} : \widehat{\mathcal{S}}_k^{\mathrm{ext}} \boxtimes \widehat{\mathcal{S}}_{n-k}^{\mathrm{ext}} \to K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})$$

categorifying parabolic induction.

What this gives:

- A categorified Zelevinsky tensor product of 2-representations.

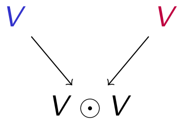
Why it matters:

- It allows us to build new affine 2-representations from smaller ones.

Example ($n = 2$, $k = 1$)

- Let V be the trivial birepresentation of $\widehat{\mathcal{S}}_1^{\mathrm{ext}} = \langle \mathrm{B}_\rho \rangle$.
- Algebra object for V: $\mathrm{X} \in \widehat{\mathcal{S}}_1^{\mathrm{ext},\diamond}$ is given by

$$\mathrm{X} := \coprod_{r \in \mathbb{Z}} \mathrm{B}_\rho^r.$$



Note: $\mathrm{B}_\rho \mathrm{X} \cong \mathrm{X}$.

- Induction: $\mathrm{Y} := \Psi_{1,1}(\mathrm{X} \boxtimes \mathrm{X}) \in K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext},\diamond})$, where

$$\mathrm{Y} \cong \coprod_{r \in \mathbb{Z}} \coprod_{s \in \mathbb{Z}} (\mathrm{B}_\rho \mathrm{T}_1)^r (\mathrm{T}_1^{-1} \mathrm{B}_\rho)^s.$$

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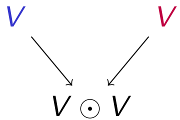

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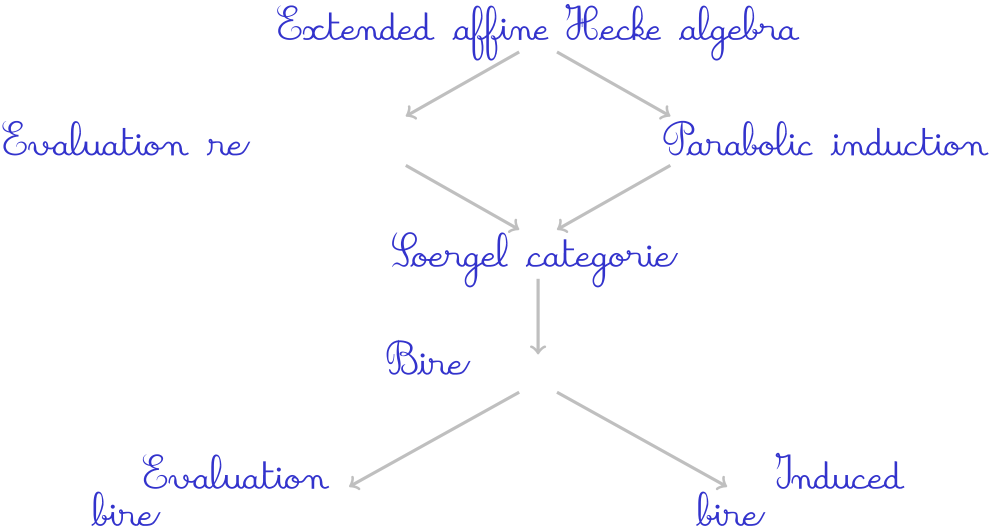
- Induced triangulated birepresentation: By a general construction due to Fan–Keller–Qiu, there is a triangulated closure of

$$\mathrm{add}\left\{ \mathrm{FY} \mid \mathrm{F} \in K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}}) \right\} \subset K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext},\diamond}).$$

Summary: what did we achieve?



Open questions and directions

- Theory of irreducibles in triangulated 2-representations.
- How can triangulated 2-reps of affine Soergel categories be classified?
- Is there a triangulated version of the Zelevinsky classification?
- Lift evaluation functor to a functor $\mathrm{Ev}^{K^b}_{r,s} \colon K^b(\widehat{\mathcal{S}}^{\mathrm{ext}}_n) \to K^b(\mathcal{S}^{\mathrm{fin}})$.
- Extension beyond affine type A.
- Connections with affine Springer theory, character sheaves, and knot homology.

Thank you!

A diagram in Rouquier–Soergel calculus